MAT 271E – Probability and Statistics

Spring 2011

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Class Meets : 13.30 – 16.30, Wednesday
EEB ?

Office Hours : 10.00 – 12.00, Wednesday

Textbook : D. B. Bertsekas, J. N. Tsitsiklis, 'Introduction to Probability', 2nd Edition, Athena-Scientific.

Supp. Text :
- S. Ross, 'Introduction to Probability and Statistics for Engineers and Scientists'.
- D. Wackerly, W. Mendenhall, R. L. Scheaffer, 'Mathematical Statistics with Applications'.

Grading : Homeworks (10%), 2 Midterms (25% each), Final (40%).

Webpage : http://web.itu.edu.tr/ibayram/Courses/MAT271E/

Tentative Course Outline

- Probability Space
  Probability models, conditioning, Bayes' rule, independence.
- Discrete Random Variables
  Probability mass function, functions of random variables, expectation, joint PMFs, conditioning, independence.
- General Random Variables
  Probability distribution function, cumulative distribution function, continuous Bayes' rule, correlation, conditional expectation.
- Limit Theorems
  Law of large numbers, central limit theorem.
- Introduction to Statistics
  Parameter estimation, hypothesis testing.

MAT 271E – Homework 1

Due 23.02.2011

1. You toss a coin repeatedly, until a 'Head' occurs. Propose a sample space for this experiment.

2. In a class, 40% of the students wear glasses, 70% wear a ring and 20% wear neither rings nor glasses. Compute the probability that a randomly chosen student

   (a) wears a ring or glasses,

   (b) wears glasses and a ring,

   (c) wears a ring but doesn't wear glasses.

3. A fair die is rolled twice and we assume that all thirty-six possible outcomes are equally likely. Let X and Y be the result of the 1st and the 2nd roll, respectively. Let A, B be events defined as,

$$A = \{X + Y \geq 9\}, \qquad B = \{\min(X, Y) \leq 4\}.$$

   Compute $\mathbf{P}(A|B)$.

4. (a) Suppose that two castles are placed randomly on a chess board. Compute the probability that they cannot capture each other.

   (b) Suppose that three castles are placed randomly on a chess board. Compute the probability that none of them can capture another.

5. Suppose that a student is successful in an exam,

   - with probability 0.3, if she eats honey for breakfast on the day of the exam and doesn't study,

   - with probability 0.9, if she studies.

   <u>Assume also that she is not successful if neither of the two conditions above are satisfied.</u> Suppose she eats honey on the day of the exam with probability 0.3. Also, suppose that with probability 0.4, she studies. Also, let 0.4 be the probability that she doesn't eat honey and doesn't study. Given that she was succesful, compute the probability that she had honey for breakfast but didn't study.

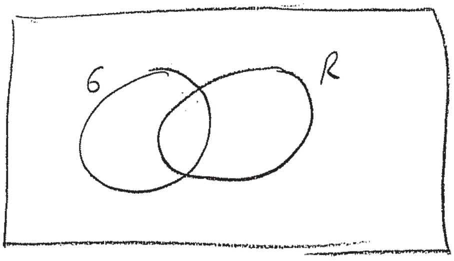
MAT271E - HW1 Solutions

1. Suppose we toss n times and end the experiment at the $(n+1)$st toss by observing a 'Head'. In this case, the output is $\underbrace{TT\dots T}_{n \text{ 'Tails'}}H$. '$n$' can be arbitrary. Therefore

$$\Omega = \{H, TH, TTH, TTTH, \dots\}. \quad (\Omega \text{ is an infinite set})$$

2. G: wears glasses
R: wears a ring

$\Rightarrow$ $G \cap R$: wears a ring **and** glasses
$G \cup R$: wears a ring or glasses
$G \cap R^c$: wears glasses but doesn't wear a ring
etc.



$\Rightarrow$
$$\left.\begin{aligned} P(G) &= 0.4 \\ P(R) &= 0.7 \\ P(R^c \cap G^c) &= 0.2 \end{aligned}\right\} \text{Given}$$

(a) $P(R \cup G) = 1 - P((R \cup G)^c) = 1 - P(R^c \cap G^c) = 0.8$

(b) $P(G \cap R) = P(G) + P(R) - P(R \cup G) = 0.4 + 0.7 - 0.8 = 0.3$

$\left(\text{Since} \quad P(R \cup G) + P(G \cap R) = P(R) + \underbrace{P(G \cap R^c) + P(G \cap R)}_{P(G)}\right)$

(c) $P(R \cap G^c) = P(R) - P(R \cap G) = 0.7 - 0.3 = 0.4$

(3.) $\Omega = \left\{ \begin{matrix} (1,1), \dots & (1,6) \\ (2,1) & \\ \vdots & \\ (6,1) \dots & (6,6) \end{matrix} \right\} \Rightarrow$ 36 equally likely outcomes.

$$A \cap B = \{ (3,6), (6,3), (4,5), (5,4), (4,6), (6,4) \} \Rightarrow P(A \cap B) = \frac{6}{36}$$

$$B^c = \{ (5,5), (5,6), (6,5), (6,6) \} \Rightarrow P(B) = 1 - P(B^c) = \frac{32}{36}$$

$$P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{6}{32}$$

(4.) (a) Suppose the first castle lands on position (i,j) (i.e. the ith row and jth column). Then, there remains 63 positions. The two castles cannot capture each other if the second castle lands on (k,l) with $i \neq k$, $j \neq l$. There are $7 \times 7 = 49$ such positions $\Rightarrow$ Desired probability $= \frac{49}{63}$

(b) Define the events

$A_2 = \{$ First two castles do not capture each other $\}$

$A_3 = \{$ The three " $\}$

Given A_2, the first two castles are on positions (i,j), (k,l) with $i \neq k$, $j \neq l$. If the third castle lands on (m,n) with $m \neq i$, $m \neq k$, $n \neq j$, $n \neq l \Rightarrow A_3$ occurs

$$P(A_3 | A_2) = \frac{6 \times 6}{62} \; ; \quad P(A_2) = \frac{49}{63} \Rightarrow P(A_3) = P(A_3|A_2) \cdot P(A_2)$$

$$= \frac{7^2 \cdot 6^2}{63 \times 62}$$

5. Define the events

$A = \{$ She eats honey $\}$

$B = \{$ She studies $\}$

$C = \{$ She is succesful $\}$.

$\Rightarrow A \cap B^c = \{$ has honey and doesn't study $\}$

Remark: C occurs only when $A \cup B$ occurs

$$\left.\begin{array}{l} P(A) = 0.3 \\ P(B) = 0.4 \\ P(A^c \cap B^c) = 0.4 \end{array}\right\} \text{Given.}$$

$$P(C) = P(C \mid (A \cap B^c)) \cdot P(A \cap B^c) + P(C \mid B) \cdot P(B) = 0.3 \times 0.2 + 0.9 \times 0.4$$

Desired probability is:

$$P(A \cap B^c \mid C) = \frac{P(A \cap B^c \cap C)}{P(C)} = \frac{P(C \mid A \cap B^c) \cdot P(A \cap B^c)}{P(C)}$$

$$= \frac{0.3 \times 0.2}{0.3 \times 0.2 + 0.9 \times 0.4}$$

MAT 271E – Homework 2

Due 02.03.2011

1. There are 8 pairs of shoes in a cabinet (a total of 16 shoes). We pick 8 shoes randomly. What is the probability that there is <u>at least</u> 1 complete pair?

2. A gambler tosses a coin three times. Each time a 'Head' comes, he gains 1 TL and each time a 'Tail' comes he loses 2 TL. Suppose that the coin is biased and the probability of 'Heads' is p. Assume also that the tosses are independent. Compute the probability that he doesn't lose money.

3. Suppose that two events A and B are independent. Let B^c denote the complement of B. Show that A and B^c are also independent. (Use the definition of independence and the probability axioms.)

4. A gambler has two coins in his pocket. One of them is fair and the other is biased so that the probability of 'Heads' is 0.6. He randomly picks one of the coins and starts tossing. Assume that the tosses are independent.

   (a) Suppose that the result of the first toss is a 'Head'. What is the probability that the coin in the gambler's hand is biased?

   (b) Suppose that the result of the first two tosses are both 'Heads'. What is the probability that the coin in the gambler's hand is biased?

5. You roll a fair die 20 times. Assume that the rolls are independent.

   (a) What is the probability that exactly 8 of the rolls turn out to be 6?

   (b) What is the probability that exactly 7 of the rolls turn out to be 6 and 9 of the rolls turn out to be odd?

6. We randomly select 9 cards from a 52-card deck.

   (a) What is the probability of selecting exactly 3 aces?

   (b) What is the probability of selecting exactly 3 aces, or exactly 2 kings, or both?

MAT271E - HW2 Solutions

(1.) If $B = \{\text{at least 1 complete pair}\}$,

$B^c = \{\text{no complete pairs}\}$.

$P(B^c)$ is easier to compute. Notice that $P(B) = 1 - P(B^c)$.

If the k^{th} pair is denoted by $\{kA, kB\}$, for B^c to occur, we need to choose either kA or kB for $k = 1, 2, .. 8$.

There are 2^8 such choices.

Total number of different combinations $= \binom{16}{8}$

$$\Rightarrow P(B^c) = \frac{2^8}{\binom{16}{8}} \Rightarrow P(B) = 1 - \frac{2^8}{\binom{16}{8}}$$

(2.) Sample space $\Omega = \{HHH, HHT, \dots\}$ $\rightarrow$ 8 different outcomes.

$A = \{\text{he doesn't lose}\} = \{HHH, HHT, HTH, THH\}$

or, $A = \{3 \text{ Heads}\} \cup \{2 \text{ Heads}\}$

$$P(A) = \binom{3}{3} p^3 + \binom{3}{2} p^2(1-p) = p^3 + 3p^2(1-p).$$

(3.) If A & B are indep $\Rightarrow P(A \cap B) = P(A) \cdot P(B)$

$P(A \cap B) + P(A \cap B^c) = P(A)$ $\rightarrow$ 2nd Axiom of Prob.

$$\Rightarrow P(A \cap B^c) = P(A) - P(A \cap B) = P(A) - P(A)P(B) = P(A) \cdot [1 - P(B)]$$
$$= P(A) P(B^c).$$

(4.) Let $A = \{\text{coin biased}\}$. Notice $P(A) = 1/2$ (any of the coins can be selected with equal prob.)

(a) $B = \{\text{first toss} = H\}$

$$P(A|B) = \frac{P(B|A) \cdot P(A)}{P(B)} \quad ; \quad P(B|A) = 0.6 \ \text{(Given)}$$

$$P(B) = P(B|A) \cdot P(A) + P(B|A^c) \cdot P(A^c) = \frac{0.6}{2} + \frac{0.5}{2}$$

$(A^c = \{\text{unbiased coin}\})$

$$\Rightarrow P(A|B) = \frac{0.3}{0.55}$$

(b) $C = \{\text{first two tosses} = HH\}$

$$P(A|C) = \frac{P(C|A) \cdot P(A)}{P(C|A) \cdot P(A) + P(C|A^c) \cdot P(A^c)} \quad ; \quad P(C|A) = (0.6)^2$$

$$P(A|C) = \frac{(0.6)^2 \times 0.5}{(0.6)^2 \times 0.5 + (0.5)^2 \times 0.5}$$

(Notice: $> P(A|B)$)

(5.) (a) $A = \{\text{outcome} = 6\} \Rightarrow P(A) = 1/6$

$A^c = \{\text{outcome} \neq 6\} \Rightarrow P(A^c) = 5/6$

$$P(\{\text{Exactly eight outcomes} = 6\}) = \binom{20}{8} \left(\frac{1}{6}\right)^8 \cdot \left(\frac{5}{6}\right)^{12}$$

(5b) Let $B = \{\text{outcome} = \text{odd}\}$ Notice $A \cap B = \emptyset$. $P(B) = 1/2$

Let $C = \{\text{exactly seven outcomes} = 6\} \cap \{9 \text{ odd outcomes}\}$

A typical element of C looks like:

$$\underbrace{666 \cdots 6}_{\text{seven } 6\text{'s}} \quad \underbrace{o_1 o_2 \cdots o_9}_{\Downarrow} \quad \underbrace{d_1 d_2 d_3 d_4}_{d_i \in P = \{2,4\}} \qquad P(D) = 1/3$$

$o_i \in O = \{1,3,5\}$ $\qquad P(O) = 1/2$

Probability of this sequence $= \left(\frac{1}{6}\right)^7 \cdot \left(\frac{1}{2}\right)^9 \cdot \left(\frac{1}{3}\right)^4$

There are $\binom{20}{7} \cdot \binom{13}{9}$ such sequences (Notice: $\binom{20}{7} \cdot \binom{13}{9} = \binom{20}{9} \cdot \binom{11}{7}$)

$$\Rightarrow P(C) = \binom{20}{7}\binom{13}{9} \cdot \left(\frac{1}{6}\right)^7 \left(\frac{1}{2}\right)^9 \left(\frac{1}{3}\right)^4$$

(6.) (a) Let $C = \{\text{Exactly 3 aces in 9 cards}\}$.

We can form an element of C by a 2-stage experiment:

1st stage: Select 3 aces out of 4 aces $\Rightarrow \binom{4}{3}$ different outcomes

2nd stage: Select 6 non-aces out of 48 non-aces $\Rightarrow \binom{48}{6}$ "

$\Rightarrow C$ has $\binom{4}{3} \cdot \binom{48}{6}$ elements. Total number of outcomes $= \binom{52}{9}$

Since cards are randomly selected, each subset is equally likely,

$$P(C) = \binom{4}{3}\binom{48}{6} \Big/ \binom{52}{9}$$

(b) Reasoning similarly, we obtain this probability as:

$$\frac{\binom{4}{3}\binom{48}{6} + \binom{4}{2}\binom{48}{7} - \binom{4}{2}\binom{4}{3}\binom{44}{4}}{\binom{52}{9}}$$

6 (b) Let $B = \{$ exactly 2 kings in 9 draws $\}$.

We are asked to compute $P(A \cup B)$.

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

Notice $P(B)$ can be computed similarly to $P(A)$ as,

$$P(B) = \binom{4}{2}\binom{48}{7} \Big/ \binom{52}{9}$$

Now, $A \cap B = \{$ exactly 2 kings <u>and</u> 3 aces in 9 draws $\}$.

This can be obtained from a 3-stage experiment.

1^{st} stage: Pick 2 kings out of 4 $\Rightarrow \binom{4}{2}$ different outcomes

2^{nd} stage: Pick 3 aces out of 4 $\Rightarrow \binom{4}{3}$ " "

3^{rd} stage: Pick 4 non-aces, non-kings out of 44 $\Rightarrow \binom{44}{4}$ d.o.

$$\Rightarrow P(A \cap B) = \binom{4}{2}\binom{4}{3}\binom{44}{4} \Big/ \binom{52}{9}$$

$$\Rightarrow P(A \cup B) = \frac{\binom{4}{3}\binom{48}{6} + \binom{4}{2}\binom{48}{7} - \binom{4}{2}\binom{4}{3}\binom{44}{4}}{\binom{52}{9}}$$

MAT 271E – Homework 3

Due 09.03.2011

1. A gambler tosses a coin three times. Each time a 'Head' comes, he gains 1 TL and each time a 'Tail' comes he loses 2 TL. Suppose that the coin is biased and the probability of 'Heads' is p. Assume also that the tosses are independent. Let X denote the amount of money (in TL) that the gambler gains (we say that he gains a negative amount if he loses money).
   (a) Find the PMF for X.
   (b) Compute $\mathbb{E}(X)$.

2. A gambler tosses a fair coin until the first 'Head' appears. If the number of tosses is n, she receives 2^n TL.
   (a) What is the expected gain of the gambler?
   (b) Suppose the gambler was asked to pay 3 TL to play this game once. She will decide to play if her probability of winning money is greater than than her probability of losing money. Will she play?

3. Suppose there are 20 boxes and in one of them is a ring. The ring is equally likely to be in any of the boxes. You open the boxes in any order you like until you find the ring. What is the expected number of boxes you open?

4. A gambler tosses a coin until the first 'Head' appears. Suppose the coin is biased and the probability of a 'Head' is p. Assume that the tosses are independent. Let X be the number of tosses. Find the mean and variance of X.

5. A troubled rabbit takes a test of 10 questions. Suppose it is known from past experience that the expected number of correct answers is 0.3.
   (a) Let A be the event defined as $A = \{\text{all answers are wrong}\}$. Find a lower bound for $P(A)$, i.e., find some $p > 0$ such that $P(A) \geq p$.
   (b) Let B be the event defined as $B = \{2 \text{ or more answers are correct}\}$. Find an upper bound for $P(B)$ that is less than $1 - p$, i.e., find some $q < (1 - p)$ such that $P(B) \leq q$.

MAT271E - HW3 Solutions

1. (a) If $A_1 = \{TTT\}$ occurs, gain $= -6$ TL

If $A_2 = \{TTH, THT, HTT\} \Rightarrow$ gain $= -3$ TL

If $A_3 = \{THH, HTH, HHT\} \Rightarrow$ gain $= 0$ TL

If $A_4 = \{HHH\} \Rightarrow$ gain $= 3$ TL

$$P(A_1) = (1-p)^3 \quad P(A_2) = p(1-p)^2 \quad P(A_3) = p^2(1-p) \quad P(A_4) = p^3$$

$$p_X(k) = \begin{cases} (1-p)^3 & \text{if } k = -6 \\ p(1-p)^2 & \text{if } k = -3 \\ p^2(1-p) & \text{if } k = 0 \\ p^3 & \text{if } k = 3 \end{cases}$$

(b) $\mathbb{E}(X) = -6 \cdot (1-p)^3 - 3p^2(1-p) + 3p^3$

2. (a) $P(\{\text{she receives } 2^n\}) = P(\{\text{'}n-1\text{' Tails, followed by 1 Head}\}) = \left(\frac{1}{2}\right)^n$.

If $X =$ gain, $p_X(k) = \left(\frac{1}{2}\right)^n$ if $k = 2^n$ for $n \in \{1, 2, 3, \dots\}$.

$$\Rightarrow \mathbb{E}(X) = \sum_{n=0}^{\infty} 2^n \cdot \left(\frac{1}{2}\right)^n = \sum_{n=0}^{\infty} 1 \longrightarrow \infty$$

(b) $P(\{\text{win money}\}) = P(\{\text{number of tosses} \geq 2\}) = \sum_{n=2}^{\infty} \left(\frac{1}{2}\right)^n = \frac{1}{2}$

(Actually, she cannot decide in this case.)

3. Notice that $P(\{\text{You find the ring in the } 1^{st} \text{ box}\}) = \frac{1}{20}$

$P(\{\text{" } 2^{nd} \text{ "}\}) = \frac{19}{20} \cdot \frac{1}{19} = \frac{1}{20}$

$\vdots$

$P(\{\text{You find it in the } k^{th} \text{ box}\}) = \frac{19}{20} \cdot \frac{18}{19} \cdots \frac{1}{20-k+1}$

$= \frac{1}{20}$

Therefore if $X = \#$ of boxes you open,

$$p_X(x) = \frac{1}{20} \quad \text{for } x \in \{1, 2, 3, \dots, 20\}.$$

$$\mathbb{E}(X) = 1 \cdot \frac{1}{20} + 2 \cdot \frac{1}{20} + \dots + 20 \cdot \frac{1}{20} = \frac{1}{20} \cdot \sum_{k=1}^{20} k = \frac{20 \cdot 21}{20 \cdot 2} = 10.5$$

(4.) Notice that $\{X = n\} = \{$'$n-1$' Tails followed by a 'Head'$\}$

so, $P(\{X = n\}) = (1-p)^{n-1} \cdot p$

$$\Rightarrow p_X(n) = (1-p)^{n-1} \cdot p \quad \text{for } n \in \{1, 2, \dots\}$$ the set of positive integers.

(This is called a geometric r.v. — see the book for an alternative solution to this problem)

$$\mathbb{E}(X) = \sum_{n=1}^{\infty} n \cdot p_X(n) = \sum_{n=1}^{\infty} n \cdot p \cdot (1-p)^{n-1} = p \cdot \sum_{n=1}^{\infty} n \cdot (1-p)^{n-1}.$$

Consider $\sum_{n=1}^{\infty} (1-p)^n$. This is a function of 'p', Let's denote it $f(p)$. If we differentiate wrt. p, provided the infinite sum converges (we will assume that), we have

$$\frac{d f(p)}{dp} = -\sum_{n=1}^{\infty} n \cdot (1-p)^{n-1}. \quad \text{But } f(p) = (1-p) \cdot \frac{1}{1-(1-p)} = \frac{1}{p} - 1$$

$$\Rightarrow \sum_{n=1}^{\infty} n (1-p)^{n-1} = \frac{1}{p^2} \Rightarrow \mathbb{E}(X) = \frac{1}{p}$$

To compute the variance we need $\mathbb{E}(X^2) = \sum_{n=1}^{\infty} n^2 (1-p)^{n-1} \cdot p$

Differentiate $f(p)$ twice $\Rightarrow \frac{2}{p^3} = \sum_{n=1}^{\infty} n(n-1)(1-p)^{n-2}$

Thus, $\frac{2(1-p)}{p^3} = \sum_{n=1}^{\infty} n^2 (1-p)^{n-1} - \underbrace{\sum_{n=1}^{\infty} n (1-p)^{n-1}}_{= -\frac{1}{p^2}}$

$\Rightarrow \mathbb{E}(X^2) = \frac{2}{p^2} - \frac{1}{p} \quad \Rightarrow var(X) = \mathbb{E}(X^2) - [\mathbb{E}(X)]^2 = \frac{1}{p^2} - \frac{1}{p}$.

5. We will only make use of the given information: if X = number of correct answers $\Rightarrow \mathbb{E}(X) = 0.3$.

$$\mathbb{E}(X) = 0 \cdot p_X(0) + 1 \cdot p_X(1) + 2 \cdot p_X(2) + \dots + 10 \cdot p_X(10).$$

(a) $P(A) = p_X(0) = 1 - \underbrace{(p_X(1) + p_X(2) + \dots + p_X(10))}_{c}$

Notice that $c \leq \mathbb{E}(X) = 0.3 \Rightarrow P(A) \geq 1 - 0.3 = 0.7$

(b) $P(B) = p_X(2) + p_X(3) + \dots + p_X(10)$

Notice $2 \cdot P(B) = 2 \cdot p_X(2) + 2 \cdot p_X(3) + \dots + 2 \cdot p_X(10) \leq \mathbb{E}(X) = 0.3$

$\Rightarrow P(B) \leq 0.15$.

MAT 271E – Homework 4

Due 14.03.2011

1. Assume that the probability of a certain player winning a game of checkers is 0.6. Let X be the number of wins in n games. What is the PMF of X? What is the expected number of wins if the player plays n games?

2. We roll a fair die with four faces, twice. Assume that the rolls are independent. Let O_1 and O_2 denote the outcomes of the first and the second roll respectively. Also, let $X = \min(O_1, O_2)$ and $Y = \max(O_1, O_2)$.

   (a) Find the marginal PMFs of X and Y.

   (b) Find the joint PMF of X and Y.

3. Suppose that the chance of a new gambler winning a game is p. Assume that the games are independent until the gambler's first win. However, once the gambler wins a game, the playing partners start to pay more attention to him and the probability of him winning decreaes to $p/2$. Assume that the games following the gambler's first win are also independent.

   (a) What is the expected number of games that the gambler needs to play to win once?

   (b) What is the expected number of games that the gambler needs to play to win twice?

4. A collector saves the stamps from the envelopes of the letters he receives. Assume that there are 100 different stamps and each one of them is equally likely to appear on an envelope that the collector receives. Assuming also that the mails are independent, what is the expected number of letters that the collector should receive in order to collect all the 100 different stamps?

5. Let X and Y be discrete independent random variables. Also let $g(\cdot)$, $h(\cdot)$ be real-valued functions. Show that $Z_1 = g(X)$ and $Z_2 = h(Y)$ are also independent.

6. We roll a fair die twice. Assume that the rolls are independent. Let X and Y denote the outcomes of the first and the second roll respectively. Also, let $Z = XY$. Compute the mean and variance of Z.

MAT 271E – HW4 Solutions

1. $p_X(k) = P\{X=k\} = \binom{n}{k}(0.6)^k(0.4)^{n-k}$

We define $X_j = \begin{cases} 1 & \text{if the player wins the } j^{th} \text{ game} \\ 0 & \text{" loses "} \end{cases}$

for $j = 1, 2, \ldots, n$.

Notice that $X = X_1 + X_2 + \ldots + X_n$

So, $\mathbb{E}(X) = \mathbb{E}(X_1 + \ldots + X_n) = \mathbb{E}(X_1) + \ldots + \mathbb{E}(X_n)$.

But $\mathbb{E}(X_j) = 0.6 \Rightarrow \mathbb{E}(X) = n \cdot 0.6$.

2. (a) $p_X(1) = P(\{X=1\}) = P(\{(1,1), (1,2), (1,3), (1,4),$
$(2,1), (3,1), (4,1)\}) = 7/16$

$p_X(2) = P(\{X=2\}) = P(\{(2,2), (2,3), (2,4), (3,2), (4,2)\}) = 5/16$

$p_X(3) = P(\{X=3\}) = P(\{(3,3), (3,4), (4,3)\}) = 3/16$

$p_X(4) = P(\{X=4\}) = P(\{(4,4)\}) = 1/16$

p_Y can be obtained from p_X, by reordering: (since $\max(O_1, O_2) = -\min(-O_1, -O_2)$)

$p_Y(1) = p_X(4) = 1/16$; $p_Y(2) = p_X(3) = 3/16$; $p_Y(3) = p_X(2) = 5/16$

$p_Y(4) = p_X(1) = 7/16$.

An alternative, and more systematic derivation of p_X is as follows:

Notice that $P(\{X \geq i\}) = P(\{O_1 \geq i \text{ and } O_2 \geq i\}) \underset{\text{since rolls are indep.}}{=} \dfrac{(4-i+1)^2}{16}$

But for $i<4$, $P(\{X=i\}) = P(\{X\geq i+1\}) - P(\{X\geq i\})$

$$= \frac{(4-i)^2-(4-i+1)^2}{16} = \frac{8-2i+1}{16}$$

and $P(\{X=4\}) = P(\{X\geq 4\}) = 1/16$.

(b) Notice that $Y \geq X$ for all outcomes.

Also, if $(X,Y)=(i,j)$ for $i\neq j$ then either $(O_1,O_2)=(i,j)$
or $(O_1,O_2)=(j,i)$

If $(X,Y)=(i,i) \Rightarrow (O_1,O_2)=(i,i)$.

So,

X \ Y	1	2	3	4
1	$1/16$	$2/16$	$2/16$	$2/16$
2	0	$1/16$	$2/16$	$2/16$
3	0	0	$1/16$	$2/16$
4	0	0	0	$1/16$

(3) (a) Let X = number of games played to win once.

Notice that $P(\{X=n\}) = (1-p)^{n-1}\cdot p$

Thus from HW3, Q4, $\mathbb{E}(X) = \frac{1}{p}$.

(b) Let Z = number of games played to win twice.

We need to find $\mathbb{E}(Z)$.

If we define Y = number of games played, after the first win, to win for a second time.

then $Z = Y+X \Rightarrow \mathbb{E}(Z) = \mathbb{E}(Y) + \mathbb{E}(X)$.

But $P(\{Y=n\}) = (1-p/2)^{n-1} \cdot p/2$

$$\Rightarrow \mathbb{E}(Y) = \frac{1}{p/2} = \frac{2}{p} \quad \Rightarrow \mathbb{E}(Z) = \frac{3}{p}.$$

(4.) Suppose the collector has collected j different stamps. Let X_j denote the number of letters he receives until he gets a new stamp that is not in his collection. Notice that $P(\{X_j = n\}) = \left(\frac{j}{100}\right)^{n-1} \cdot \left(1 - \frac{j}{100}\right)$.

Therefore $\mathbb{E}(X_j) = \dfrac{100}{100-j}$

But if Z = number of letters to collect all the 100 different stamps,

then $Z = \underset{\substack{\Downarrow \\ X_0}}{1} + X_1 + \dots + X_{99}$

$$\Rightarrow \mathbb{E}(Z) = 1 + \sum_{j=1}^{99} \mathbb{E}(X_j) = 1 + \sum_{j=1}^{99} \frac{100}{100-j} = 1 + \sum_{j=1}^{99} \frac{100}{j}$$

(5.) Let us show this in two steps. First,

<u>Lemma</u>: If $A_1, A_2, \dots, A_n$ are disjoint events that are independent from another event B then $\left(\bigcup_{i=1}^{n} A_i\right)$ is independent from B as well.

<u>Pf</u>: $$P\left(\left(\bigcup_{i=1}^{n} A_i\right) \cap B\right) \underset{\substack{\text{2nd} \\ \text{axiom of} \\ \text{prob.}}}{=} \sum_{i=1}^{n} P(A_i \cap B) \underset{\text{independence}}{=} \sum_{i=1}^{n} P(B) \cdot P(A_i)$$

$\Rightarrow$

$$P\left(\left(\bigcup_{i=1}^{n} A_i\right)\cap B\right) = P(B)\sum_{i=1}^{n} P(A_i) = P(B)\cdot P\left(\bigcup_{i=1}^{n} A_i\right)$$

<u>end of proof</u>

We assume for simplicity that the ranges of X and Y are finite sets.

<u>Prop</u>: If X has a finite range, and X & Y are independent, then $g(X)$ and Y are also indep.

<u>Pf</u>: Define the sets $X_\ell = \{x \in X : g(x) = \ell\}$.

Notice that for $a \in X_\ell$, $\{X=a\}$ and $\{Y=k\}$ are independent by assumption. Therefore by the lemma, X_ℓ and $\{Y=k\}$ are also independent and thus follows the proof.

To see that $g(X)$ and $h(Y)$ are also independent, apply the proposition to $Z = g(X)$ (independent of Y) and $h(Y)$.

6. By independence, $\mathbb{E}(Z) = \mathbb{E}(X)\cdot\mathbb{E}(Y) = \left(\sum_{i=1}^{6} \frac{1}{6}\cdot i\right)^2 = (3.5)^2$

and $\mathbb{E}(Z^2) = \mathbb{E}(X^2)\cdot\mathbb{E}(Y^2) = \left[\sum_{i=1}^{6} \frac{1}{6}\cdot(i)^2\right]^2 = \left(\frac{6\cdot 7\cdot 13}{6}\right)^2\cdot\left(\frac{1}{6}\right)^2$

$$\Rightarrow \operatorname{var}(Z) = \mathbb{E}(Z^2) - \left[\mathbb{E}(Z)\right]^2 = \left(\frac{7\cdot 13}{6}\right)^2 - \left(\frac{7}{2}\right)^4$$

MAT 271E – Homework 5

Due 30.03.2011

1. Let X be uniformly distributed on the interval [-3,2]. We define a discrete random variable Y as,
$$Y = \begin{cases} 1 & \text{if } X \geq 0, \\ 0 & \text{if } X < 0. \end{cases}$$
What is the PMF of Y?

2. You just missed a bus and you're waiting for the next one. Suppose that the probability that the next bus arrives in T units of time (for a non-negative T!) is given by
$$\int_0^T c\,e^{-c\,t}\,dt$$
where 'c' is a constant. Let X be the amount of time you wait for the next bus to arrive.
   (a) What is the probability density function (pdf) of X?
   (b) What is the expected amount of time you need to wait?
   (c) What is the variance of X?

3. The taxis in a city are numbered from 1 to n, where n is the total number of taxis. You try to estimate n as follows. Suppose that after seeing i taxis, your estimate is E_i (and you start from $E_0 = 0$). At your next observation of a taxi, whose number is, say X_{i+1}, you set $E_{i+1} = \max(X_{i+1}, E_i)$. Assume that, any time you observe a taxi, its number is equally likely to be any one in the set $\{1, \ldots, n\}$, independent of previous observations.
   (a) What is the CDF of E_2?
   (b) For $i > 0$, what is the CDF of E_i?
   (c) For $i > 0$, what is the PMF of E_i?
   (d) Compute $\lim_{i\to\infty} \mathbb{E}(E_i)$ and $\lim_{i\to\infty} \mathrm{var}(E_i)$.

MAT271E - HW5 Solutions

(1.) Y can only take the values $0, 1$.

$$P_Y(0) = P(\{Y=0\}) = P(\{X<0\}) = \int_{-\infty}^{0} f_X(t)\,dt = \int_{-3}^{0} \frac{1}{5}\,dt = \frac{3}{5}$$

$$P_Y(1) = P(\{Y=1\}) = P(\{X\geq 0\}) = \int_{0}^{2} \frac{1}{5}\,dt = \frac{2}{5}.$$

(2.) (a) Let $s \leq u < 0$. We have that $\underbrace{P\{X\in[s,u]\}}_{=0} = \int_{s}^{u} f_X(t)\,dt$

Since $f_X(t)$ is a non-negative function,

this implies that $f_X(t) = 0$ for $t<0$.

Now, if $0 \leq s \leq u$,

$$P(\{X\in[s,u]\}) = P(\{X\leq u\}) - P(\{X\leq s\})$$

$$= \int_0^u ce^{-ct}\,dt - \int_0^s ce^{-ct}\,dt = \int_s^u ce^{-ct}\,dt$$

Therefore, by the defn. of pdf, $f_X(t) = \begin{cases} c\,e^{-ct} & \text{for } t\geq 0 \\ 0 & \text{for } t<0 \end{cases}$

(b) $$E(X) = \int_{-\infty}^{\infty} t f_X(t)\,dt = \int_0^{\infty} tce^{-ct}\,dt = -e^{-ct}\cdot t\,\Big|_0^{\infty} - \int_0^{\infty} -e^{-ct}\,dt$$

$$= \frac{1}{c}$$

(b) $\operatorname{var}(X) = \mathbb{E}(X^2) - [\mathbb{E}(X)]^2$.

$$\mathbb{E}(X^2) = \int_{-\infty}^{\infty} t^2 f_X(t)\,dt = \int_0^{\infty} t^2 c e^{-ct}\,dt = -e^{-ct} t^2 \Big|_0^{\infty} - \int_0^{\infty} -e^{-ct} \cdot 2t\,dt$$

$$= \frac{2}{c}\int_0^{\infty} c \cdot e^{-ct} \cdot t\,dt = \frac{2}{c^2} \quad (\text{from part (a)})$$

$$\Rightarrow \operatorname{var}(X) = \frac{1}{c^2}.$$

3. Notice $p_{X_i}(k) = \frac{1}{n}$ for $1 \le k \le n$. $\Rightarrow P(\{X_i \le k\}) = \frac{k}{n}$.

if k is an integer, and, $1 \le k \le n$.

(a) $E_2 = \max(X_1, X_2)$.

$F_{E_2}(k) = P(\{E_2 \le k\})$. But notice $\{E_2 \le k\}$ is the same event as $\{X_1 \le k\} \cap \{X_2 \le k\}$. Since X_1 and X_2 are independent, we have $F_{E_2}(k) = \frac{k^2}{n^2}$ if k is an integer, and $1 \le k \le n$.

If t is not an integer, define $\lfloor t \rfloor$ = integer part of t.

$$\Rightarrow F_{E_2}(t) = \begin{cases} \frac{\lfloor t \rfloor^2}{n^2} & \text{if } 1 \le t < n+1, \\ 1 & \text{if } t \ge n+1 \\ 0 & \text{if } t < 1 \end{cases}$$

(b) Reasoning similarly as in (a), notice that, $E_i = \max(X_1, X_2, \dots, X_i)$.

$$\{E_i \le t\} = \{X_1 \le t\} \cap \{X_2 \le t\} \cap \dots \cap \{X_i \le t\}.$$

So,
$$F_{E_i}(t) = \begin{cases} \dfrac{(\lfloor t \rfloor)^i}{n^i} & \text{if } 1 \le t < n+1 \\ 1 & \text{if } t \ge n+1 \\ 0 & \text{if } t < 1 \end{cases}$$

(c) Since E_i can only take integer values, between 1 and n,

$$P(\{E_i = k\}) = P(\{E_i \le k\}) - P(\{E_i \le k-1\}).$$
$$\Downarrow \qquad\qquad \Downarrow \qquad\qquad \Downarrow$$
$$p_{E_i}(k) = F_{E_i}(k) - F_{E_i}(k-1) = \frac{k^i}{n^i} - \frac{(k-1)^i}{n^i} \quad \text{for } k = 1, \dots, n.$$

(d)
$$\lim_{i\to\infty} \mathbb{E}(E_i) = \lim_{i\to\infty} \sum_{k=1}^{n} k \cdot \left\{ \left(\frac{k}{n}\right)^i - \left[\frac{(k-1)}{n}\right]^i \right\}$$

$$= \sum_{k=1}^{n} k \cdot \left(\lim_{i\to\infty} \left(\frac{k}{n}\right)^i - \left(\frac{k-1}{n}\right)^i \right) = n$$

Similarly
$$\lim_{i\to\infty} \mathbb{E}(E_i^2) = \sum_{k=1}^{n} k^2 \left(\lim_{i\to\infty} \left(\frac{k}{n}\right)^i - \left(\frac{k-1}{n}\right)^i \right) = n^2$$

$$\Rightarrow \operatorname{var}(E_i) = \mathbb{E}(E_i^2) - [\mathbb{E}(E_i)]^2 = 0.$$

MAT 271E – Homework 6

Due 06.04.2011

1. Consider an experiment that involves an infinite number of coins being tossed at the same time. Suppose we number the coins and denote the outcome of the n^{th} toss as O_n. We define

$$X_n = \begin{cases} 1, & \text{if } O_n = \text{Head}, \\ 0, & \text{if } O_n = \text{Tail}. \end{cases}$$

We construct a number (in the base-2 system) as

$$Z = 0.X_1 X_2 X_3 \ldots$$

(a) Propose a sample space, Ω, for this experiment.

(b) Consider the events $A = \{Z \geq 1/2\}$, $B = \{Z \geq 1/4\}$. Express these events in terms of the sample space you proposed in (a).

(c) If the coin tosses are independent of each other and the probability of observing a Head for each coin is given by p, compute the proababilities of A and B.

2. Noise in communication channels at a particular instant is usually modelled as a Gaussian random variable. Let us denote it by X. Suppose for the sake of simplicity that X is a *standard* Gaussian random variable (i.e. it has zero mean and its variance is one). The power of noise is defined as $Y = X^2$. Notice that Y is also a random variable. In this question we will derive the probability distribution function (pdf) of Y. Recall that

$$\Phi(t) = P\big(\{X \leq t\}\big) = \int_{-\infty}^{t} f_X(t)\, dt,$$

where

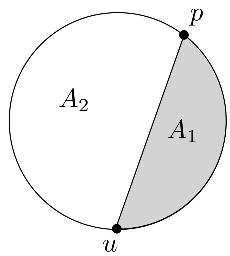
$$f_X(t) = \frac{1}{\sqrt{2\pi}}\, e^{-2t^2}.$$

(a) Express the cumulative distribution function (cdf) of Y, that is $F_Y(s) = P\big(\{Y \leq s\}\big)$ in terms of the function $\Phi(\cdot)$.

(b) Obtain the pdf of Y, denoted by $f_Y(s)$, by differentiating $F_Y(s)$ (you can write it explicitly).

3. You choose two points on a line segment independently, according to a uniform distribution and break the segment into three parts. What is the probability that you can form a triangle with the resulting segments?

4. You randomly choose two points p, u, independently, according to a uniform distribution on a circle of radius 1 as shown below.



The chord between p and u divides the circle into two segments A_1 and A_2, where A_1 is the one with smaller area. Compute the expected areas of A_1 and A_2.

5. Let X_1 be a continuous random variable, uniformly distributed on $[0, 1]$. Let Y be an exponential random variable, i.e.,

$$f_Y(t) = \begin{cases} c\, e^{-ct} & \text{if } t \geq 0, \\ 0, & \text{if } t < 0. \end{cases}$$

Compute the probability that $X \leq Y$.

MAT271E - HW6 Solutions

1. (a) $\Omega =$ All sequences of the form $(O_1, O_2, \dots)$ where O_i is one of $\{H, T\}$.

(b) $A = $ (All sequences that start with H) $\cup \{(T, \underbrace{H, H, H, \dots}_{\text{all Heads}})\}$

$B = $ (All sequences ~~that start~~ that contain at least one H in the first two positions) $\cup \{(T, T, \underbrace{H, H, H, \dots}_{\text{all Heads}})\}$

(c) $P(A) = p + (1-p) \cdot p^{\infty} = p$

$P(B) = \left[1 - (1-p)^2\right] + (1-p)^2 p^{\infty} = 1 - (1-p)^2$

2. (a) $F_Y(s) = \begin{cases} P(\{X^2 \leq s\}) = P(\{-\sqrt{s} \leq X \leq \sqrt{s}\}) = \Phi(\sqrt{s}) - \Phi(-\sqrt{s}) & \text{for } s \geq 0 \\ 0 & \text{for } s < 0 \end{cases}$

(b) $f_Y(s) = \begin{cases} \frac{\partial}{\partial s} F_Y(s) = \frac{1}{2\sqrt{s}} \cdot \Phi'(\sqrt{s}) + \frac{1}{2\sqrt{s}} \Phi'(-\sqrt{s}) = \frac{1}{\sqrt{2\pi s}} e^{-2s} & \text{for } s \geq 0 \\ 0 & \text{for } s < 0 \end{cases}$

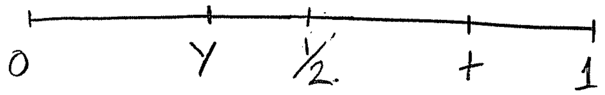
(3.) Let X, Y denote the two points.

Let $A_1 = \{X_1 > \frac{1}{2}\}$. $P(A_1) = \frac{1}{2}$.

We remark that, $P(\text{Triangle}) = P(\text{Tri} / A) \cdot P(A) + P(\text{Tri} / A^c) \cdot P(A^c)$.

Also, by symmetry, $P(\text{Tri} / A) = P(\text{Tri} / A^c)$, so $P(\text{Tri}) = P(\text{Tri} / A)$.

Let us compute $P(\text{Tri} / A)$. Notice that given A, X is uniformly distributed on $[\frac{1}{2}, 1]$. $\left(f_{X|A}(t) = \begin{cases} 2 & \text{for } t \in [\frac{1}{2}, 1] \\ 0 & \text{otherwise} \end{cases} \right)$



Assume $X = t$. Then we can form a triangle if and only if

$$\{Y < \tfrac{1}{2}\} \cap \{0 < t - Y < \tfrac{1}{2}\} = \{t - \tfrac{1}{2} < Y < \tfrac{1}{2}\}$$

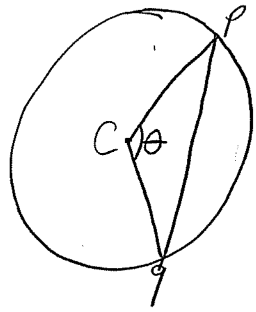
happens. The probability of this event is $= \frac{1}{2} - \left(\frac{1}{2} - t\right) = 1 - t$.

Therefore, $P(\text{Tri} / A) = \int_{1/2}^{1} 2 \cdot (1-t)\, dt = 1 - \left(t^2 \Big|_{1/2}^{1}\right) = \frac{1}{4}$

$\Rightarrow P(\text{Triangle}) = \frac{1}{4}$.

(See the book for an alternative solution.)

(4) Let C be the center and θ be defined as



θ is uniformly distributed on $[0, \pi]$.

Moreover given θ, Area of $A_1 = \frac{\theta}{2} - \underbrace{\frac{\sin\theta}{2}}_{}$ ⟶ Area of triangle

$\Rightarrow$ Expected area of A_1 $= \frac{1}{\pi}\int_0^{\pi} \left(\frac{\theta}{2} - \frac{\sin\theta}{2}\right) d\theta = \frac{1}{\pi} \cdot \left(\frac{\pi^2}{4} + \frac{\cos\theta}{2}\Big|_0^{\pi}\right)$

$$\frac{\pi}{4} + \frac{1}{\pi}\left(-\frac{1}{2} - \frac{1}{2}\right) = \frac{\pi}{4} - \frac{1}{\pi}$$

Since $(\text{Area of } A_2) = \pi - (\text{Area of } A_1)$

$\mathbb{E}(\text{Area of } A_2) = \pi - \left(\frac{\pi}{4} - \frac{1}{\pi}\right) = \frac{3\pi}{4} + \frac{1}{\pi}$

(5) $P(\{Y \geq X\} \mid \{X = s\}) = \int_s^{\infty} c e^{-ct} dt = -e^{-ct}\Big|_s^{\infty} = e^{-sc}$

$\Rightarrow P(\{Y \geq X\}) = \int_0^1 e^{-sc} \underbrace{f_X(s)}_{=1} ds = -\frac{1}{c} e^{-sc}\Big|_0^1 = \frac{1}{c}(1 - e^{-c})$

MAT 271E – Homework 7

Due 13.04.2011

1. We are given a biased coin and we are told that the probability of Heads is P. Assume that P is a random variable whose pdf is

$$f_P(t) = \begin{cases} 2t & \text{if } t \in [0,1], \\ 0 & \text{if } t \notin [0,1]. \end{cases}$$

   Suppose we start tossing the coin. Assume that the tosses are independent.

   (a) What is the probability that the first toss is a Head?

   (b) Given that the first toss is a Head, compute the conditional pdf of P.

   (c) Given that the first toss is a Head, compute the probability that the second toss is also a Head.

2. Let X and Y be independent random variables, uniformly distributed on $[0,1]$. Find the pdf of $Z = X/Y$.

3. Let X and Y be independent random variables, uniformly distributed on $[0,1]$. Find the pdf of $Z = X + Y$.

4. Let X and Y be independent random variables, uniformly distributed on $[0,1]$. Find the pdf of $Z = |X - Y|$.

5. Let X be a standard Gaussian random variable. Compute $\mathbb{E}(X^3)$ and $\mathbb{E}(X^4)$.

MAT271E – HW7 Solutions

① (a) $P(\{H\}) = \int P(\{H\} \mid \{P=t\}) \, f_p(t)\,dt = \int_0^1 t \cdot 2t \, dt = \frac{2}{3}.$

(b) Let $A = \{\text{Toss is a Head}\}$. Let $0 \le a \le b \le 1$.

$$P(\{a \le P \le b\} \mid A) = \frac{P(A \cap \{a \le P \le b\})}{P(A)}$$

$$= \frac{\int_a^b (t \cdot 2t)\,dt}{2/3} = \int_a^b 3t^2\,dt = \int_a^b f_{P|A}(t)\,dt$$

$$\Rightarrow f_p(t) = \begin{cases} 3t^2 & \text{for } t \in [0,1] \\ 0 & \text{for } t \notin [0,1] \end{cases}$$

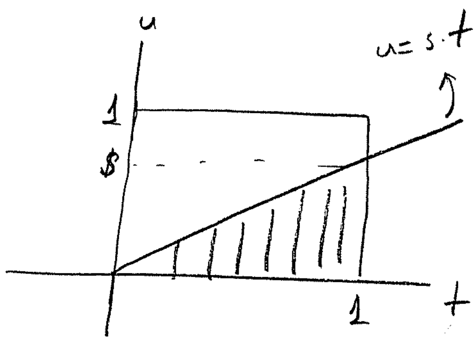
(c) $P(\underbrace{\{\text{Two consecutive Heads}\}}_{B}) = \int P(B \mid \{P=t\}) \cdot f_p(t)\,dt$

$$= \int_0^1 t^2 \, 2t \, dt = \frac{2}{4}.$$

$$\Rightarrow P(\{2^{nd} \text{ Toss} = H\} \mid \{1^{st} \text{ Toss} = H\}) = \frac{P(B)}{P\{1^{st} \text{ Toss} = H\}} = \frac{1}{2} \cdot \frac{3}{2} = \frac{3}{4}.$$

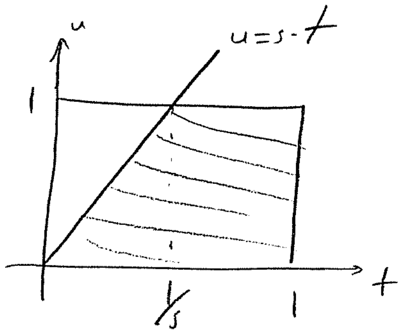
Notice this can also be computed using $f_{P|A}(t)$ as: $\int_0^1 t \cdot 3t^2\,dt = \frac{3}{4}.$

(2.) $f_{X,Y}(t,u) = \begin{cases} 1 & \text{if } 0 \le t \le 1 \text{ and } 0 \le u \le 1 \\ 0 & \text{otherwise} \end{cases}$



$\underbrace{P(X/Y \le s)}_{\substack{\shortparallel \\ X \le tY}}$ is equal to the area of the shaded triangle for $s \le 1$

$= \frac{s}{2}$



For $s \ge 1$ $\quad P(X/Y \le s) \Rightarrow$

$= 1 - \frac{1}{2s}$

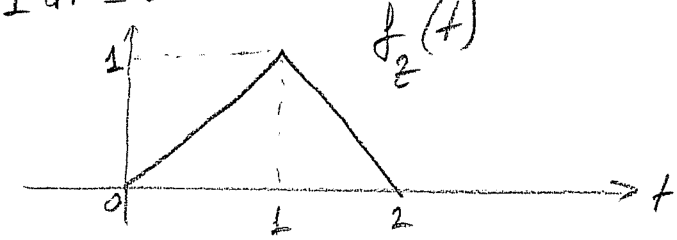
$$\Rightarrow F_Z(s) = \begin{cases} 0 & \text{for } s \le 0 \\ s/2 & \text{for } 0 < s \le 1 \\ 1 - 2/s & \text{for } 1 < s \end{cases} \Rightarrow f_Z(s) = \begin{cases} 0 & \text{for } s \le 0 \\ 1/2 & \text{for } 0 < s \le 1 \\ 2/s^2 & \text{for } 1 < s \end{cases}$$

(3.) $F_Z(s) = P(Z \le s) = \int \underbrace{P(Y + t \le s)}_{= F_Y(s-t)} \cdot f_X(t)\, dt$

$\Rightarrow f_Z(s) = \int f_Y(s-t) f_X(t)\, dt$

If $s \in [0,1] \Rightarrow f_Z(s) = \int_0^s 1 \cdot 1 \, dt = s$ $\qquad$ If $s \notin [0,2] \Rightarrow f_Z(s) = 0$

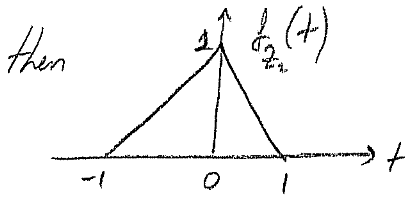
If $s \in [1,2] \Rightarrow f_Z(s) = \int_{s-1}^{1} 1 \cdot 1 \, dt = 2 - s$

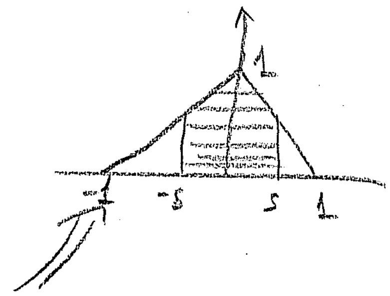


4. Let $Z_1 = -Y \Rightarrow f_{Z_1}(t) = \begin{cases} 1 & \text{if } t \in [-1, 0] \\ 0 & \text{otherwise} \end{cases}$

It can be shown, similar to Q3 that if $Z_2 = X + Z_1$

then





Finally $Z = |Z_2|$.

$\Rightarrow$ For $s \in [0,1]$, $P\{Z \leq s\} = \int_{-s}^{s} f_{Z_2}(t)\,dt = 1 - (1-s)^2$

$= 2s - s^2$

$$\Rightarrow f_Z(s) = \begin{cases} 2 - 2s & \text{for } s \in [0,1] \\ 0 & \text{for } s \notin [0,1] \end{cases}$$

5. Recall that the moment generating function for the standard Gaussian is: $M(s) = e^{s^2/2}$

$$\mathbb{E}(X^3) = \frac{d^3}{ds^3} M(s)\Big|_{s=0}, \quad \mathbb{E}(X^4) = \frac{d^4}{ds^4} M(s)\Big|_{s=0}$$

$M'(s) = s e^{s^2/2}$; $M''(s) = e^{s^2/2} + s^2 e^{s^2/2}$; $M'''(s) = s e^{s^2/2}(1+s^2) + 2s e^{s^2/2}$

$= M(s)(1+s^2)$ $\qquad = e^{s^2/2}(3s + s^3)$

$$\frac{d^4}{ds^4} M(s) = e^{s^2/2}(3s^2 + s^4) + e^{s^2/2}(3 + 3s^2)$$

$\Rightarrow \mathbb{E}(X^3) = 0$ (In fact all odd moments of st. Gaus $= 0$ - Why?

$\Rightarrow \mathbb{E}(X^4) = 3$

MAT 271E – Homework 8

Due 04.05.2011

1. We are given independent, identically distributed (iid) observations $X_1, X_2, \ldots, X_n$ which have Gaussian distributions with mean 3, and unknown variance v.

   (a) Find the maximum likelihood estimate of v.

   (b) Is the estimate you found biased? If it is biased, can you propose an unbiased estimator?

   (c) Is the estimate you found in part (a) consistent?

2. We are given independent, identically distributed (iid) observations $X_1, X_2, \ldots, X_n$ which are uniformly distributed on the interval $[0, \theta]$.

   (a) Find the maximum likelihood estimate of θ.

   (b) Is the estimate you found biased? If it is biased, can you propose an unbiased estimator?

   (c) Is the estimate you found in part (a) consistent?

3. Consider a biased die where the probability of observing a '1' is equal to θ. Let k be a fixed number. You roll the die until you observe k 1's. Let N be the number of rolls. Assuming that the rolls are independent, find the ML estimator of θ based on N.

MAT 271E – Probability and Statistics

Midterm Examination I

16.03.2011

Student Name : ____________________

Student Num. : ____________________

5 Questions, 120 Minutes

Please Show Your Work!

(10 pts) 1. Consider an experiment that consists of a coin toss, followed by rolling a die.

(a) Propose a sample space for this experiment.

(b) Assuming that each possible outcome is equally likely, compute the probability of the event

$$A = \{\text{the toss is a ‘Head’ and the roll is less than or equal to 4}\}.$$

(30 pts) 2. In a quiz, a single question will be asked about one of the three distinct subjects, namely A, B, or C. A certain student will pass the quiz with probability

- 2/3 if the question is about A,
- 2/3 if the question is about B,
- 1/3 if the question is about C.

Assume that the question is about

- A with probability 1/4,
- B with probability 1/2.

(a) Compute the probability that the student passes.

(b) Given that the student has passed, compute the probability that the question was on B.

(c) Given that the student has failed, compute the probability that the question was not on B.

(20 pts) 3. Suppose there are 4 red and 4 blue balls in an urn. You randomly draw balls from the urn, without putting them back into the urn, until you draw the first red ball. Let us define X to be the number of balls drawn from the urn.

(a) Find the probability mass function (PMF) of X.

(b) Consider the event

$$A = \{\text{at least two draws are made}\}.$$

Find the conditional PMF of X given A.

(15 pts) 4. Let X and Y be random variables whose joint PMF is given by the table below.

		X			
		3	4	5	6
Y	-1	3/18	1/18	0	2/18
	0	2/18	0	0	3/18
	1	2/18	1/18	0	4/18

(a) Find the marginal PMF of X.

(b) Let us define a new random variable as $Z = X + Y$. Find the PMF of Z.

(25 pts) 5. Sermet and Sevgi are playing backgammon. Suppose that the couple plays n games. Assume that, initially, the probability of Sermet winning a game is p. However, with each game, Sevgi learns more about the game so that at the k^{th} game, the probability that Sermet wins decreases to p^k. Assume that the games are independent. Let X denote the number of games that Sermet wins.

(a) Compute the expected value of X.

(b) Compute the variance of X.

MAT 271E – Probability and Statistics

Midterm Examination II

20.04.2011

Student Name : ____________________

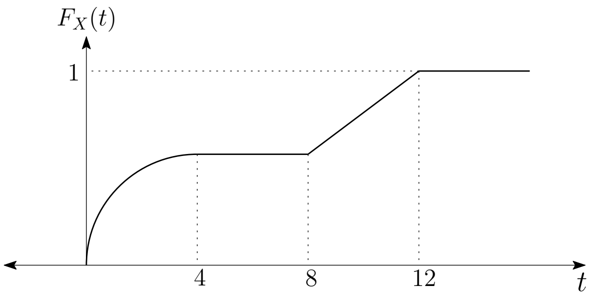
Student Num. : ____________________

5 Questions, 90 Minutes

Please Show Your Work!

(15 pts) 1. Consider a random variable, X, whose cumulative distribution function (cdf) is,

$$F_X(t) = \begin{cases} 0 & \text{for } t < 0, \\ \frac{1}{2}\sin\left(\frac{\pi}{8}t\right) & \text{for } 0 \le t < 4, \\ \frac{1}{2} & \text{for } 4 \le t \le 8, \\ \frac{1}{8}t - \frac{1}{2} & \text{for } 8 \le t \le 12, \\ 1 & \text{for } 12 \le t, \end{cases}$$



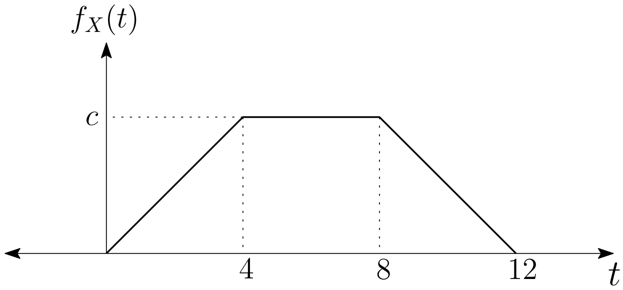
as shown on the right.

Let us define the events $A = \{2 \le X \le 10\}$ and $B = \{6 \le X \le 12\}$.

(a) Compute $P(A)$, the probability of the event A.

(b) Compute $P(B|A)$, the conditional probability of the event B, given A.

(15 pts) 2. Consider a random variable, X, whose probability density function (pdf) is shown below.



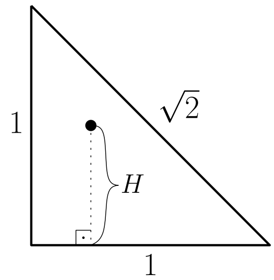
Let us define the events $A = \{2 \le X \le 10\}$ and $B = \{6 \le X \le 12\}$.

(a) Find c.

(b) Compute $P(A)$, the probability of the event A. (In terms of 'c', if you do not have an answer to part (a).)

(c) Compute $P(B|A)$, the conditional probability of the event B, given A.

(25 pts) 3. Consider the triangle below. Suppose we choose a point randomly inside the triangle (i.e. any point inside the triangle is equally likely to be chosen). Let H denote the vertical distance of the point to the base of the triangle.



(a) Specify $f_H(t)$, the probability distribution function (pdf) of H.

(b) Compute $\mathbb{E}(H)$, the expected value of H.

(20 pts) 4. Utku sells umbrellas. On a rainy day, he makes 'X' TL where X is a random variable uniformly distributed on $[40, 80]$. If it doesn't rain, he cannot make any money. Assume that the probability that it rains on a given day is equal to $1/5$. Assume also that the probability that it rains, or the money he makes on a certain day is independent of whatever happens on any other day. Compute the expected amount of money that Utku makes in 30 days.

Please explain your reasoning for full credit.

(25 pts) 5. Let X and Y be independent random variables, uniformly distributed on $[0, 1]$. Let $Z = XY$.

(a) Specify $F_Z(t)$, the cumulative distribution function (cdf) of Z.

(b) Specify $f_Z(t)$, the probability density function (pdf) of Z.

MAT 271E – Probability and Statistics

Final Examination

02.06.2011

Student Name : ____________________

Student Num. : ____________________

5 Questions, 120 Minutes

Please Show Your Work!

(15 pts) 1. Consider an experiment that consists of the roll of a fair die (a six-face, regular die).

(a) Propose a sample space, Ω, for this experiment.

(b) Consider the event $A = \{\text{the outcome is even}\}$. What is the probability of A?

(c) Given the event $A = \{\text{the outcome is even}\}$, compute the conditional probability of the event $B = \{\text{the outcome is not } 4\}$.

(d) Suppose we define a discrete random variable X as,

$$X = \begin{cases} 1, & \text{if the outcome is strictly less than 5,} \\ 2, & \text{if the outcome is strictly greater than 4.} \end{cases}$$

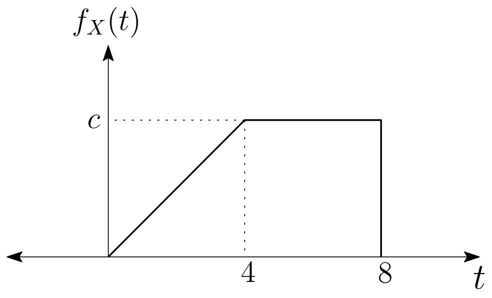
Find $P_X(k)$, the probability mass function (PMF) of X.

(20 pts) 2. A gambler has two coins in his pocket. One of them is fair and the other is biased. For the biased coin, the probability of 'Heads' is 3/4. He randomly picks and tosses one of the coins (probability of picking either of the coins is 1/2).

(a) Compute the probability that the toss is a 'Head'.

(b) Given that the toss is a 'Head', what is the probability that the coin in the gambler's hand is biased?

(25 pts) 3. Consider a random variable, X, whose probability density function (pdf) is shown below.



Let us define the events $A = \{2 \leq X \leq 6\}$ and $B = \{1 \leq X \leq 4\}$.

(a) Find c.

(b) Compute $P(A)$, the probability of the event A. (In terms of 'c', if you do not have an answer to part (a).)

(c) Compute $P(B|A)$, the conditional probability of the event B, given A.

(d) Compute $\mathbb{E}(X)$, the expected value of X.

(e) Determine and sketch the cumulative distribution function (cdf) of X.

(20 pts) 4. Let Y be a random variable, uniformly distributed on $[0, X]$. Let Z be a random variable, uniformly distributed on $[0, X^2]$.

(a) For $X = 1/2$, compute the probability of the event $\{Y \leq Z\}$.

(b) Assume that X is a random variable, uniformly distributed on $[0, 1]$. Compute the probability of the event $\{Y \leq Z\}$.

(20 pts) 5. Consider two independent random variables, X, Y, that are uniformly distributed on $[0, \theta]$. Using X and Y, we define a new random variable as $Z = \max(X, Y)$. Notice that the event $\{Z \leq t\}$ can be expressed as,

$$\{Z \leq t\} = \{X \leq t\} \cap \{Y \leq t\}.$$

(a) Compute the probability of the event $\{Z \leq t\}$ in terms of θ. Find $F_Z(t)$, the cumulative distribution function (cdf) of Z.

(b) Find $\mathbb{E}(Z)$, the expected value of Z.

(c) Suppose θ is an unknown parameter of interest. A student proposes to use Z as an estimator for θ. Is Z a biased or an unbiased estimator for θ? If it is biased, can you propose an unbiased estimator?

(d) Biased or not, we decide to use Z as the estimator for θ. Find the value of c so that the interval $[Z, Z + c]$ contains θ with probability 99/100.
