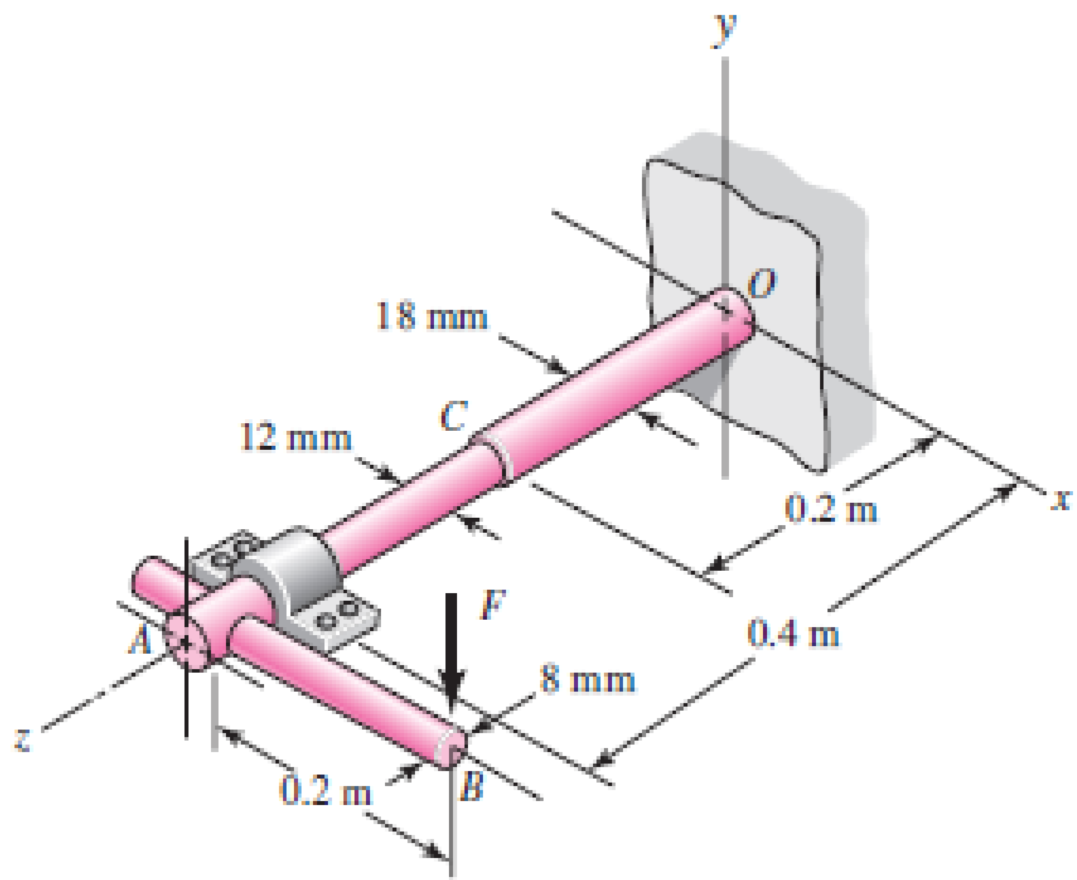
The figure illustrates a stepped torsion-bar spring *OA* with an actuating cantilever *AB*. Both parts are of carbon steel. Use superposition and find the spring rate *k* corresponding to a force *F* acting at *B*.



From Table A-5, E = 207 GPa, and G = 79.3 GPa.

$$|y_B| = \left(\frac{Tl}{GJ}\right)_{OC} l_{AB} + \left(\frac{Tl}{GJ}\right)_{AC} l_{AB} + \frac{Fl_{AB}^{3}}{3EI_{AB}} = \frac{Fl_{OC}l_{AB}^{2}}{G\left(\pi d_{OC}^{4}/32\right)} + \frac{Fl_{AC}l_{AB}^{2}}{G\left(\pi d_{AC}^{4}/32\right)} + \frac{Fl_{AB}^{3}}{3E\left(\pi d_{3}^{4}/64\right)}$$

$$= \frac{32Fl_{AB}^{2}}{\pi}\left[\frac{l_{OC}}{Gd_{OC}^{4}} + \frac{l_{AC}}{Gd_{AC}^{4}} + \frac{2l_{AB}}{3Ed_{AB}^{4}}\right]$$

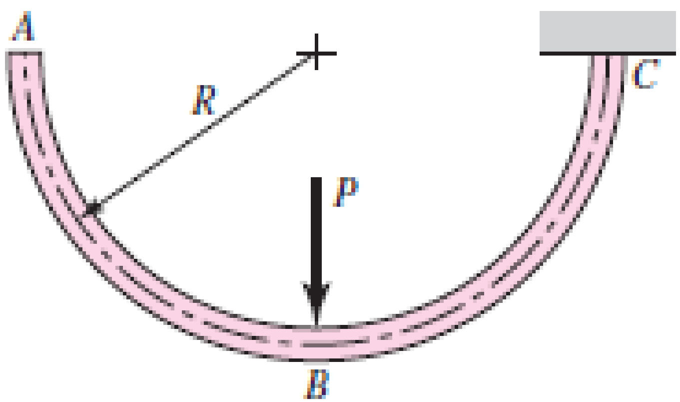
The spring rate is $k = F/\,|y_B|$. Thus

$$k = \left\{\frac{32l_{AB}^{2}}{\pi}\left[\frac{l_{OC}}{Gd_{OC}^{4}} + \frac{l_{AC}}{Gd_{AC}^{4}} + \frac{2l_{AB}}{3Ed_{AB}^{4}}\right]\right\}^{-1}$$

$$= \left\{\frac{32\left(200^{2}\right)}{\pi}\left[\frac{200}{79.3\left(10^{3}\right)18^{4}} + \frac{200}{79.3\left(10^{3}\right)12^{4}} + \frac{2(200)}{3(207)10^{3}\left(8^{4}\right)}\right]\right\}^{-1}$$

$$= 8.10 \text{ N/mm} \qquad \textit{Ans.}$$

The steel curved bar shown has a rectangular cross section with a radial height h = 6 mm, and a thickness b = 4 mm. The radius of the centroidal axis is R = 40 mm. A force P = 10 N is applied as shown. Find the vertical deflection at B. Use Castigliano's method for a curved flexural member, and since $R/h < 10$, do not neglect any of the terms.



There is no bending in AB. Using the variable θ, rotating counterclockwise from B

$$M = PR\sin\theta \qquad \frac{\partial M}{\partial P} = R\sin\theta$$

$$F_r = P\cos\theta \qquad \frac{\partial F_r}{\partial P} = \cos\theta$$

$$F_\theta = P\sin\theta \qquad \frac{\partial F_\theta}{\partial P} = \sin\theta$$

$$\frac{\partial MF_\theta}{\partial P} = 2PR\sin^2\theta$$

$A = 6(4) = 24\ \text{mm}^2$, $r_o = 40 + \frac{1}{2}(6) = 43$ mm, $r_i = 40 - \frac{1}{2}(6) = 37$ mm,

From Table 3-4, p.121, for a rectangular cross section

$$r_n = \frac{6}{\ln(43/37)} = 39.92489\ \text{mm}$$

From Eq. (4-33), the eccentricity is $e = R - r_n = 40 - 39.92489 = 0.07511$ mm

From Table A-5, $E = 207(10^3)$ MPa, $G = 79.3(10^3)$ MPa

From Table 4-1, $C = 1.2$

From Eq. (4-38)

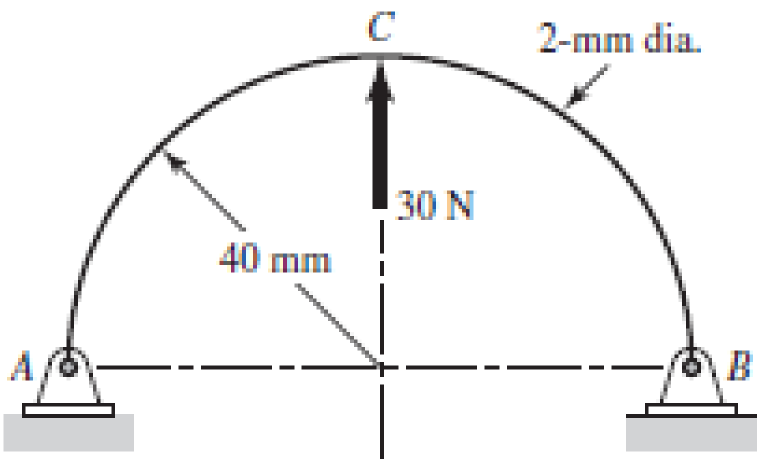
$$\delta = \int_0^{\frac{\pi}{2}} \frac{M}{AeE}\left(\frac{\partial M}{\partial P}\right)d\theta + \int_0^{\frac{\pi}{2}} \frac{F_\theta R}{AE}\left(\frac{\partial F_\theta}{\partial P}\right)d\theta - \int_0^{\frac{\pi}{2}} \frac{1}{AE}\frac{\partial(MF_\theta)}{\partial P}d\theta + \int_0^{\frac{\pi}{2}} \frac{CF_rR}{AG}\left(\frac{\partial F_r}{\partial P}\right)d\theta$$

$$= \int_0^{\frac{\pi}{2}} \frac{P(R\sin\theta)^2}{AeE}d\theta + \int_0^{\frac{\pi}{2}} \frac{PR(\sin\theta)^2}{AE}d\theta - \int_0^{\frac{\pi}{2}} \frac{2PR\sin^2\theta}{AE}d\theta + \int_0^{\frac{\pi}{2}} \frac{CPR(\cos\theta)^2}{AG}d\theta$$

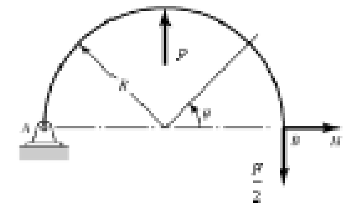
$$= \frac{\pi PR}{4AE}\left(\frac{R}{e} + 1 - 2 + \frac{EC}{G}\right) = \frac{\pi(10)(40)}{4(24)(207\cdot 10^3)}\left(\frac{40}{0.07511} + 1 - 2 + \frac{(207\cdot 10^3)(1.2)}{79.3\cdot 10^3}\right)$$

$\delta = 0.0338$ mm *Ans.*

For the steel wire form shown, use Castigliano's method to determine the horizontal reaction forces at *A* and *B* and the deflection at *C*.



$R/h = 20 > 10$ so Eq. (4-41) can be used to determine deflections. Consider the horizontal reaction, to applied at *B*, subject to the constraint $(\delta_B)_H = 0$.



$$M = \frac{FR}{2}(1-\cos\theta) - HR\sin\theta \qquad \frac{\partial M}{\partial H} = -R\sin\theta \qquad 0<\theta<\frac{\pi}{2}$$

By symmetry, we may consider only half of the wire form and use twice the strain energy Eq. (4-41) then becomes,

$$(\delta_B)_H = \frac{\partial U}{\partial H}\,B\,\frac{2}{EI}\int_0^{\pi/2}\left(M\frac{\partial M}{\partial H}\right)R\,d\theta = 0$$

$$\int_0^{\pi/2}\left[\frac{FR}{2}(1-\cos\theta) - HR\sin\theta\right](-R\sin\theta)R\,d\theta = 0$$

$$-\frac{F}{2}+\frac{F}{4}+H\frac{\pi}{4}=0 \Rightarrow H=\frac{F}{\pi}=\frac{30}{\pi}=9.55\text{ N} \qquad \textit{Ans.}$$

Reaction at *A* is the same where *H* goes to the left. Substituting *H* into the moment equation we get,

$$M = \frac{FR}{2\pi}[\pi(1-\cos\theta)-2\sin\theta] \qquad \frac{\partial M}{\partial F} = \frac{R}{2\pi}[\pi(1-\cos\theta)-2\sin\theta] \qquad 0<\theta<\frac{\pi}{2}$$

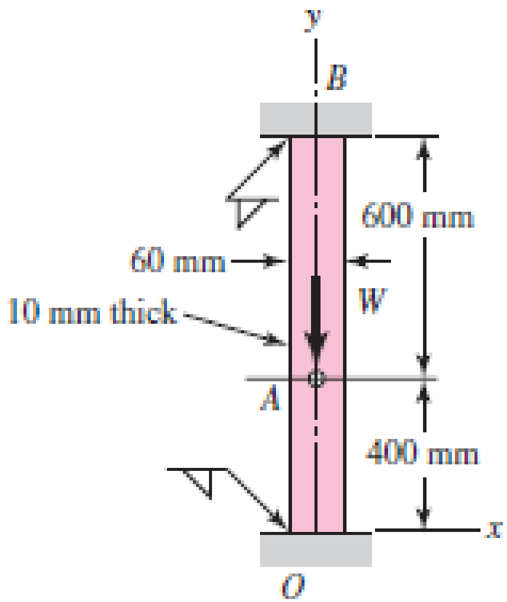
$$\delta_p = \frac{\partial U}{\partial P}\,B\int\frac{2}{EI}\left(M\frac{\partial M}{\partial F}\right)R\,d\theta = \frac{2}{EI}\int_0^{\pi/2}\frac{FR^2}{4\pi^2}[\pi(1-\cos\theta)-2\sin\theta]^2 R\,d\theta$$

$$= \frac{FR^3}{2\pi^2 EI}\int_0^{\pi/2}(\pi^2+\pi^2\cos^2\theta+4\sin^2\theta-2\pi^2\cos\theta-4\pi\sin\theta+4\pi\sin\theta\cos\theta)\,d\theta$$

$$= \frac{FR^3}{2\pi^2 EI}\left[\pi^2\left(\frac{\pi}{2}\right)+\pi^2\left(\frac{\pi}{4}\right)+4\left(\frac{\pi}{4}\right)-2\pi^2-4\pi+2\pi\right]$$

$$= \frac{(3\pi^2-8\pi-4)}{8\pi}\frac{FR^3}{EI} = \frac{(3\pi^2-8\pi-4)}{8\pi}\frac{(30)(40^3)}{207(10^3)\left[\pi(2^4)/64\right]} = 0.224\text{ mm} \qquad \textit{Ans.}$$

A rectangular aluminum bar 10 mm thick and 60 mm wide is welded to fixed supports at the ends, and the bar supports a load W = 4 kN, acting through a pin as shown. Find the reactions at the supports and the deflection of point A.



$$R_O + R_B = W \qquad (1)$$

$$\delta_{OA} = \delta_{AB}$$

$$\left(\frac{Fl}{AE}\right)_{OA} = \left(\frac{Fl}{AE}\right)_{AB}$$

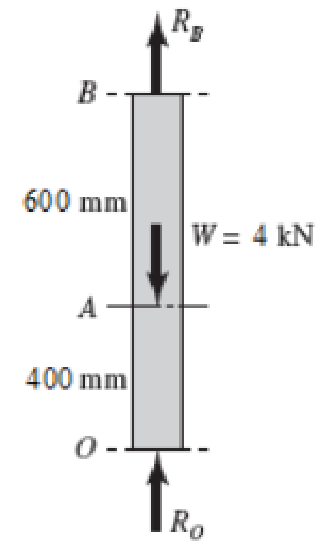
$$\frac{400R_O}{AE} = \frac{600R_B}{AE} \quad \Rightarrow \quad R_O = \frac{3}{2}R_B \qquad (2)$$

Substitute this unto Eq. (1)

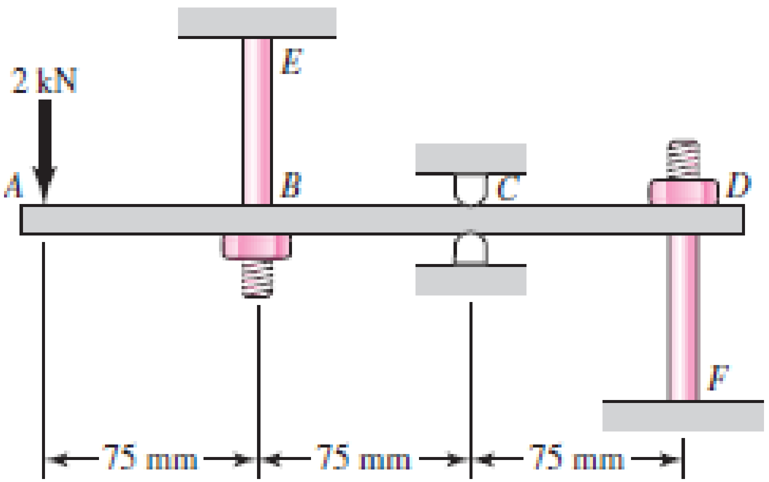
$$\frac{3}{2}R_B + R_B = 4 \quad \Rightarrow \quad R_B = 1.6 \text{ kN} \qquad \textit{Ans.}$$

From Eq. (2) $\quad R_O = \frac{3}{2}1.6 = 2.4 \text{ kN} \qquad \textit{Ans.}$

$$\delta_A = \left(\frac{Fl}{AE}\right)_{OA} = \frac{2400(400)}{10(60)(71.7)(10^3)} = 0.0223 \text{ mm} \qquad \textit{Ans.}$$



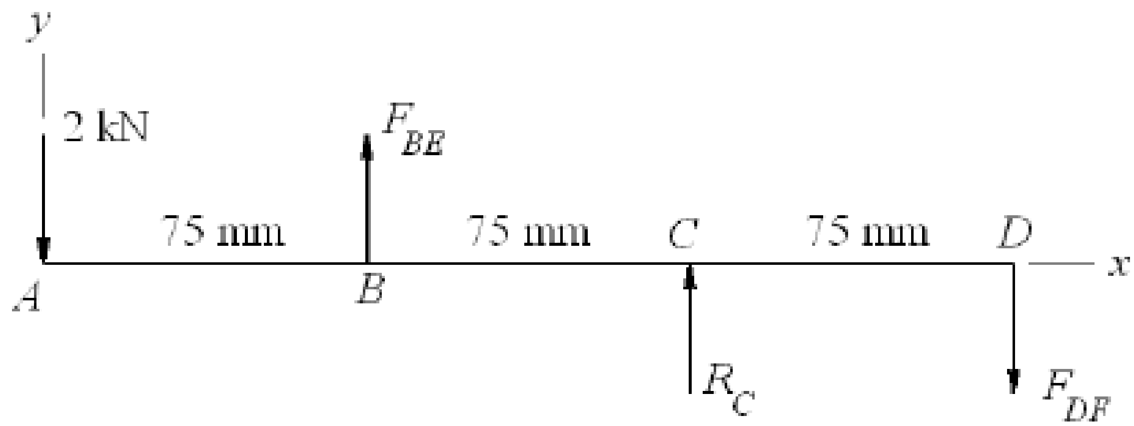
The steel beam *ABCD* shown is supported at *C* as shown and supported at *B* and *D* by shoulder steel bolts, each having a diameter of 8 mm. The lengths of *BE* and *DF* are 50 mm and 65 mm, respectively. The beam has a second area moment of 21(103) mm4. Prior to loading, the members are stress-free. A force of 2 kN is then applied at point *A.* Using procedure 2 of Sec. 4–10, determine the stresses in the bolts and the deflections of points *A*, *B*, and *D*.



Beam: $EI = 207(10^3)21(10^3)$
$= 4.347(10^9)$ N·mm^2.

Rods: $A = (\pi/4)8^2 = 50.27$ mm^2.

Procedure 2.



1. Statics.

$R_C + F_{BE} - F_{DF} = 2\,000$ (1)

$R_C + 2F_{BE} = 6\,000$ (2)

2. Bending moment equation.

$$M = -2\,000\,x + F_{BE}\langle x - 75\rangle^1 + R_C\langle x - 150\rangle^1$$

$$EI\frac{dy}{dx} = -1000x^2 + \frac{1}{2}F_{BE}\langle x-75\rangle^2 + \frac{1}{2}R_C\langle x-150\rangle^2 + C_1 \quad (3)$$

$$EIy = -\frac{1000}{3}x^3 + \frac{1}{6}F_{BE}\langle x-75\rangle^3 + \frac{1}{6}R_C\langle x-150\rangle^3 + C_1x + C_2 \quad (4)$$

3. B.C 1. At $x = 75$ mm,

$$y_B = -\left(\frac{Fl}{AE}\right)_{BE} = -\frac{F_{BE}(50)}{50.27(207)10^3} = -4.805(10^{-6})F_{BE}$$

Substituting into Eq. (4) at $x = 75$ mm,

$$4.347(10^9)\left[-4.805(10^{-6})F_{BE}\right] = -\frac{1000}{3}(75^3) + C_1(75) + C_2$$

Simplifying gives

$$20.89\left(10^3\right)F_{BE}+75C_1+C_2=140.6\left(10^6\right) \qquad (5)$$

B.C 2. At x = 150 mm, y = 0. From Eq. (4),

$$-\frac{1000}{3}\left(150^3\right)+\frac{1}{6}F_{BE}\left(150-75\right)^3+C_1\left(150\right)+C_2=0$$

or,

$$70.31\left(10^3\right)F_{BE}+150C_1+C_2=1.125\left(10^9\right) \qquad (6)$$

B.C 3. At x = 225 mm,

$$y_D=\left(\frac{Fl}{AE}\right)_{DF}=\frac{F_{DF}\left(65\right)}{50.27\left(207\right)10^3}=6.246\left(10^{-6}\right)F_{DF}$$

Substituting into Eq. (4) at x = 225 mm,

$$4.347(10^9)\left[6.246(10^{-6})F_{DF}\right]=-\frac{1000}{3}(225^3)+\frac{1}{6}F_{BE}(225-75)^3$$
$$+\frac{1}{6}R_C(225-150)^3+C_1(225)+C_2$$

Simplifying gives

$$70.31(10^3)R_C+562.5(10^3)F_{BE}-27.15(10^3)F_{DF}+225C_1+C_2=3.797(10^9) \qquad (7)$$

Equations (1), (2), (5), (6), and (7) in matrix form are

$$\begin{pmatrix} 1 & 1 & -1 & 0 & 0 \\ 1 & 2 & 0 & 0 & 0 \\ 0 & 20.89(10^3) & 0 & 75 & 1 \\ 0 & 70.31(10^3) & 0 & 150 & 1 \\ 70.31(10^3) & 562.5(10^3) & -27.15(10^3) & 225 & 1 \end{pmatrix}\begin{Bmatrix} R_C \\ F_{BE} \\ F_{DF} \\ C_1 \\ C_2 \end{Bmatrix}=\begin{Bmatrix} 2(10^3) \\ 6(10^3) \\ 140.6(10^6) \\ 1.125(10^9) \\ 3.797(10^9) \end{Bmatrix}$$

Solve simultaneously or use software. The results are

$R_C = -2378$ N, $F_{BE} = 4189$ N, $F_{DF} = -189.2$ N *Ans.*
and $C_1 = 1.036\ (10^7)$ N·mm^2, $C_2 = -7.243\ (10^8)$ N·mm^3.

The bolt stresses are $\sigma_{BE} = 4189/50.27 = 83.3$ MPa, $\sigma_{DF} = -189/50.27 = -3.8$ MPa *Ans.*

The deflections are

From Eq. (4) $\quad y_A=\dfrac{1}{4.347(10^9)}\left[-7.243(10^8)\right]=-0.167\text{ mm} \qquad$ *Ans.*

For points B and D use the axial deflection equations*.

$$y_B=-\left(\frac{Fl}{AE}\right)_{BE}=-\frac{4189(50)}{50.27(207)10^3}=-0.0201\text{ mm} \qquad \textit{Ans.}$$

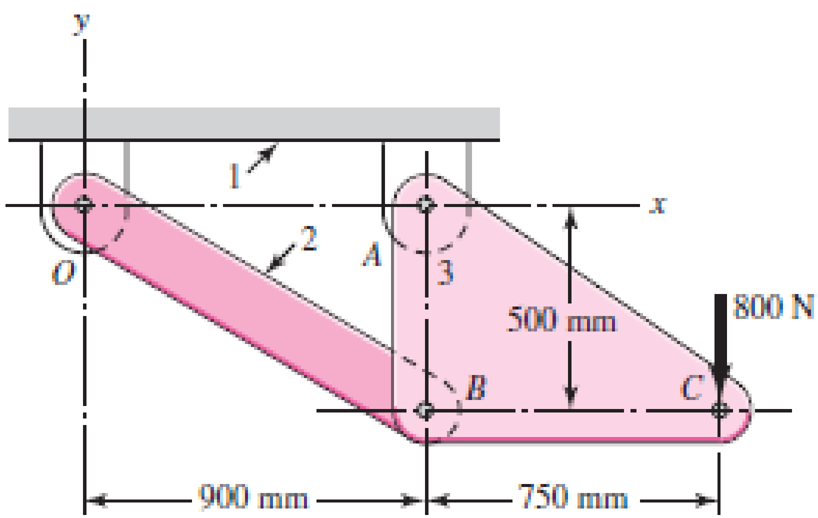
$$y_D=\left(\frac{Fl}{AE}\right)_{DF}=\frac{-189(65)}{50.27(207)10^3}=-1.18(10^{-3})\text{ mm} \qquad \textit{Ans.}$$

*Note. The terms in Eq. (4) are quite large, and due to rounding are not very accurate for calculating the very small deflections, especially for point D.

Link 2, shown in the figure, is 25 mm wide, has 12-mm-diameter bearings at the ends, and is cut from low-carbon steel bar stock having a minimum yield strength of 165 MPa. The end-condition constants are $C = 1$ and $C = 1.2$ for buckling in and out of the plane of the drawing, respectively.

(*a*) Using a design factor $n_d = 4$, find a suitable thickness for the link.

(*b*) Are the bearing stresses at *O* and *B* of any significance?



(a) $\Sigma M_A = 0, \quad (0.75)(800) - \dfrac{0.9}{\sqrt{0.9^2 + 0.5^2}} F_{BO}(0.5) = 0 \quad \Rightarrow \quad F_{BO} = 1373 \text{ N}$

Using $n_d = 4$, design for $F_{cr} = n_d F_{BO} = 4(1373) = 5492$ N

$$l = \sqrt{0.9^2 + 0.5^2} = 1.03 \text{ m}, \quad S_y = 165 \text{ MPa}$$

In-plane:

$$k = \left(\frac{I}{A}\right)^{1/2} = \left(\frac{bh^3/12}{bh}\right)^{1/2} = 0.2887h = 0.2887(0.025) = 0.007\,218 \text{ m}, \quad C = 1.0$$

$$\frac{l}{k} = \frac{1.03}{0.007218} = 142.7$$

$$\left(\frac{l}{k}\right)_1 = \left(\frac{2\pi^2(207)(10^9)}{165(10^6)}\right)^{1/2} = 157.4$$

Since $(l/k)_1 > (l/k)$ use Johnson formula.

Try 25 mm x 12 mm,

$$P_{cr} = 0.025(0.012)\left\{165(10^6) - \left[\frac{165(10^6)}{2\pi}(142.7)\right]^2 \frac{1}{1(207)10^9}\right\} = 29.1 \text{ kN}$$

This is significantly greater than the design load of 5492 N found earlier. Check out-of-plane.

Out-of-plane: $k = 0.2887(0.012) = 0.003\ 464$ in, $C = 1.2$

$$\frac{l}{k} = \frac{1.03}{0.003\ 464} = 297.3$$

Since $(l/k)_1 < (l/k)$ use Euler equation.

$$P_{cr} = 0.025(0.012)\frac{1.2\pi^2(207)10^9}{297.3^2} = 8321 \text{ N}$$

This is greater than the design load of 5492 N found earlier. It is also significantly less than the in-plane P_{cr} found earlier, so the out-of-plane condition will dominate. Iterate the process to find the minimum h that gives P_{cr} greater than the design load.

With $h = 0.010$, $P_{cr} = 4815$ N (too small)
$h = 0.011$, $P_{cr} = 6409$ N (acceptable)

Use 25 mm x 11 mm. If standard size is preferred, use 25 mm x 12 mm. *Ans.*

(b) $\sigma_b = -\dfrac{P}{dh} = -\dfrac{1373}{0.012(0.011)} = -10.4(10^6) \text{ Pa} = -10.4 \text{ MPa}$

No, bearing stress is not significant. *Ans.*