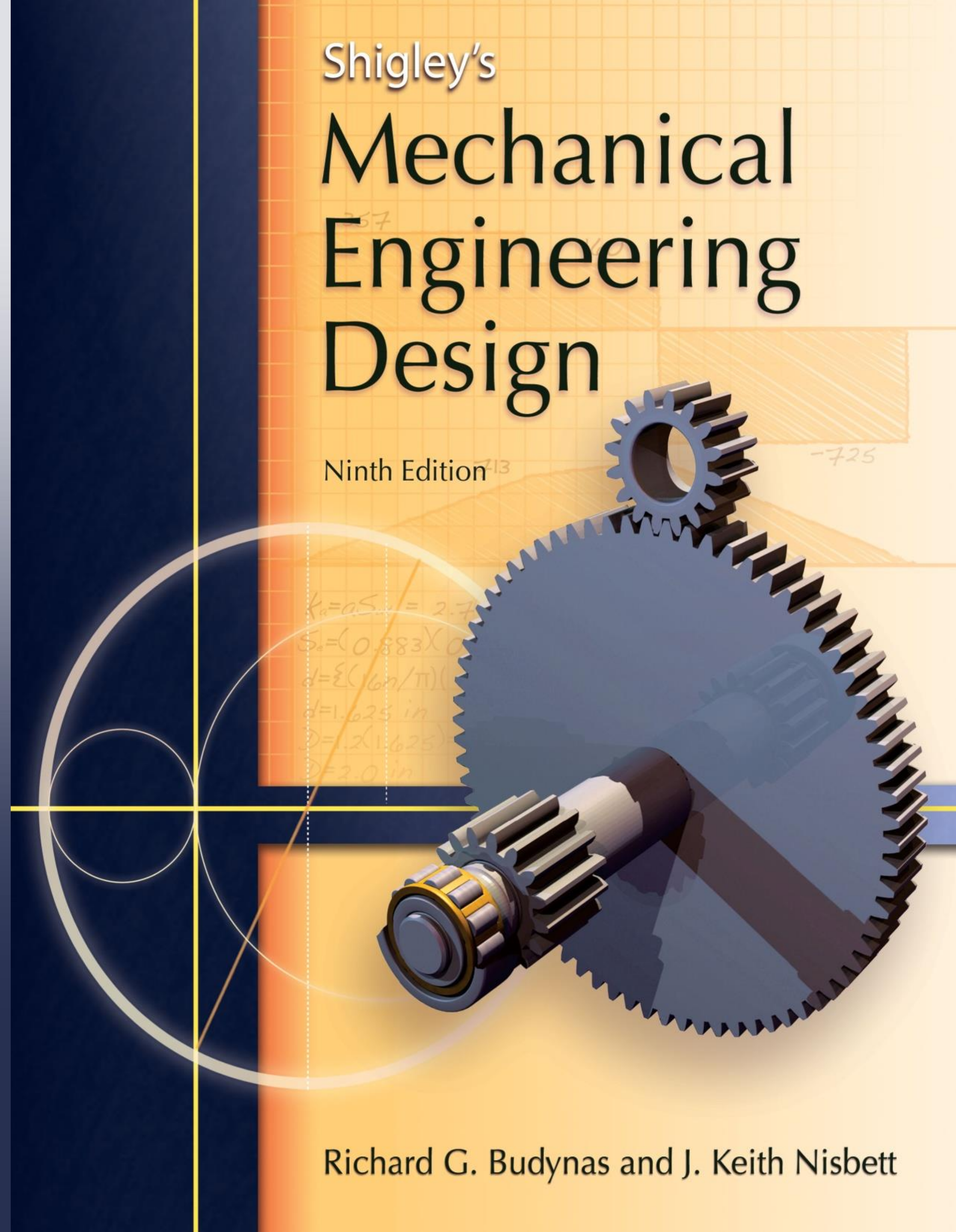
Lecture Slides

Chapter 10

Mechanical Springs



Chapter Outline

Mechanical Springs

- Exert Force
- Provide flexibility
- Store or absorb energy

Helical Spring

- Helical coil spring with round wire
- Equilibrium forces at cut section anywhere in the body of the spring indicates direct shear and torsion

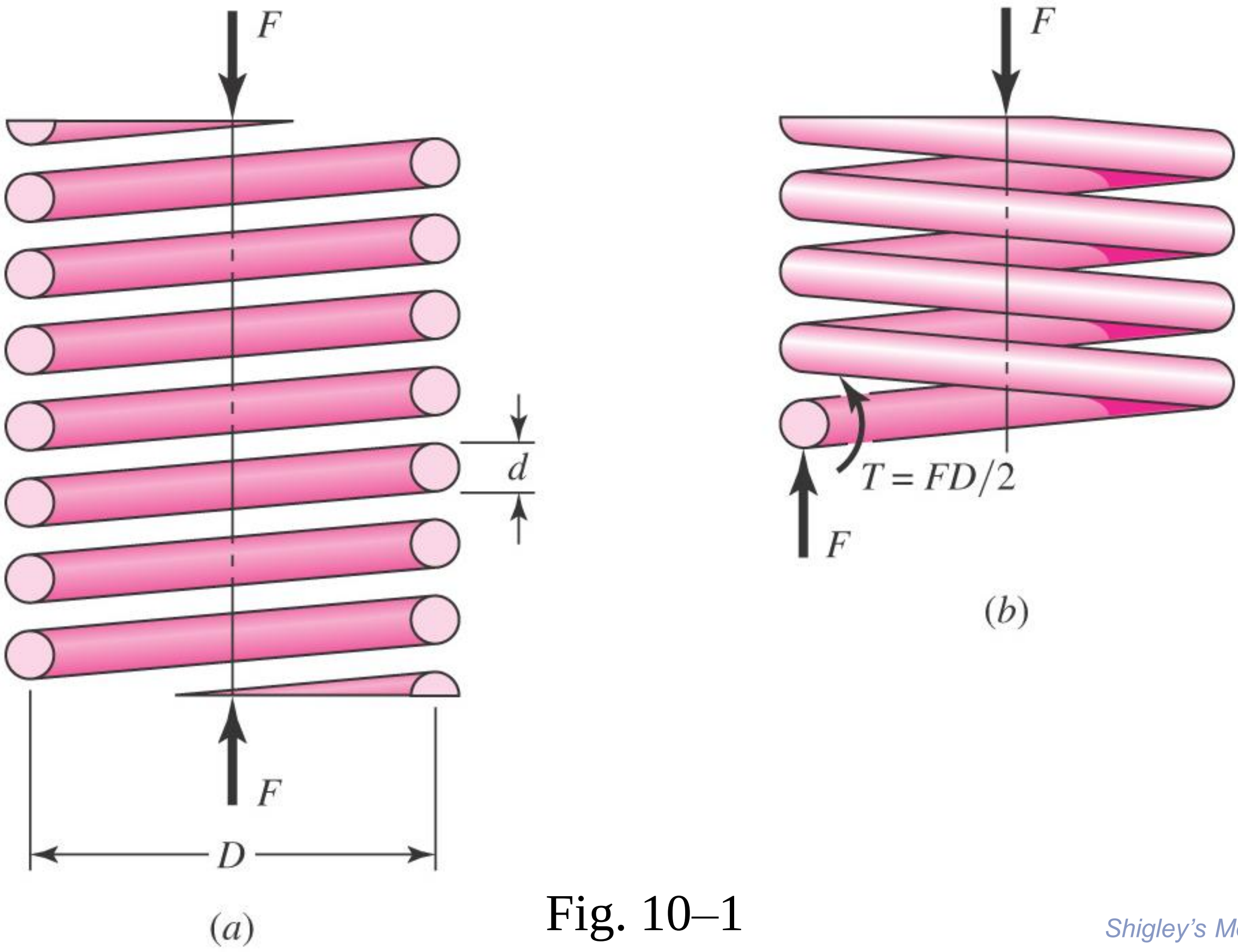


Fig. 10–1

Stresses in Helical Springs

- Torsional shear and direct shear
- Additive (maximum) on inside fiber of cross-section

$$\tau_{max} = \frac{Tr}{J} + \frac{F}{A}$$

- Substitute terms

$$\tau_{max} = \tau,\ T = FD/2,\ r = d/2,$$

$$J = \pi d^4/32 \qquad A = \pi d^2/4$$

$$\tau = \frac{8FD}{\pi d^3} + \frac{4F}{\pi d^2}$$

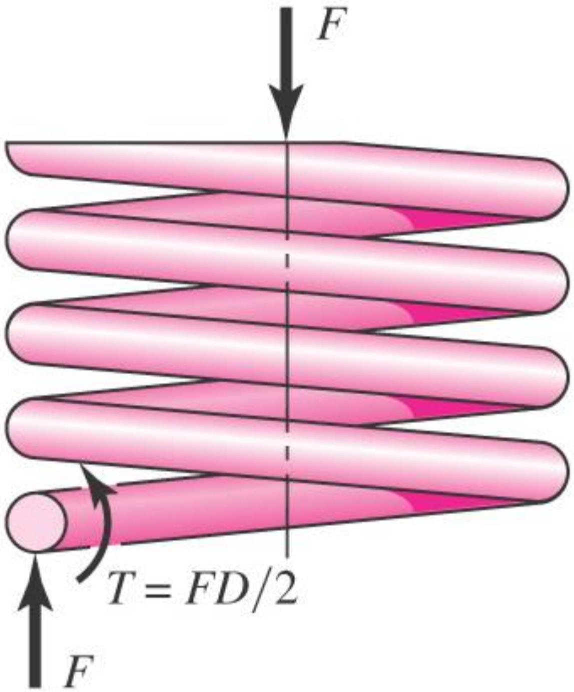


Fig. 10–1*b*

Stresses in Helical Springs

$$\tau = \frac{8FD}{\pi d^3} + \frac{4F}{\pi d^2}$$

Factor out the torsional stress

$$\tau = \left(1 + \frac{d}{2D}\right)\left(\frac{8FD}{\pi d^3}\right)$$

Define *Spring Index* $C = \frac{D}{d}$ (10–1)

Define *Shear Stress Correction Factor*

$$K_s = 1 + \frac{1}{2C} = \frac{2C+1}{2C} \quad (10\text{–}3)$$

Maximum shear stress for helical spring

$$\tau = K_s \frac{8FD}{\pi d^3} \quad (10\text{–}2)$$

Curvature Effect

- Stress concentration type of effect on inner fiber due to curvature
- Can be ignored for static, ductile conditions due to localized cold-working
- Can account for effect by replacing K_s with *Wahl factor* or *Bergsträsser factor* which account for both direct shear and curvature effect

$$K_W = \frac{4C-1}{4C-4} + \frac{0.615}{C} \qquad (10\text{–}4)$$

$$K_B = \frac{4C+2}{4C-3} \qquad (10\text{–}5)$$

$$\tau = K_B \frac{8FD}{\pi d^3} \qquad (10\text{–}7)$$

- Cancelling the curvature effect to isolate the curvature factor

$$K_c = \frac{K_B}{K_s} = \frac{2C(4C+2)}{(4C-3)(2C+1)} \qquad (10\text{–}6)$$

Deflection of Helical Springs

Use Castigliano's method to relate force and deflection

$$U = \frac{T^2 l}{2GJ} + \frac{F^2 l}{2AG}$$

Substituting $T = FD/2$, $l = \pi DN$, $J = \pi d^4/32$, and $A = \pi d^2/4$

$$U = \frac{4F^2 D^3 N}{d^4 G} + \frac{2F^2 DN}{d^2 G}$$

$$y = \frac{\partial U}{\partial F} = \frac{8FD^3 N}{d^4 G} + \frac{4FDN}{d^2 G}$$

$$y = \frac{8FD^3 N}{d^4 G}\left(1 + \frac{1}{2C^2}\right) \doteq \frac{8FD^3 N}{d^4 G} \qquad (10–8)$$

$$k \doteq \frac{d^4 G}{8D^3 N} \qquad (10–9)$$

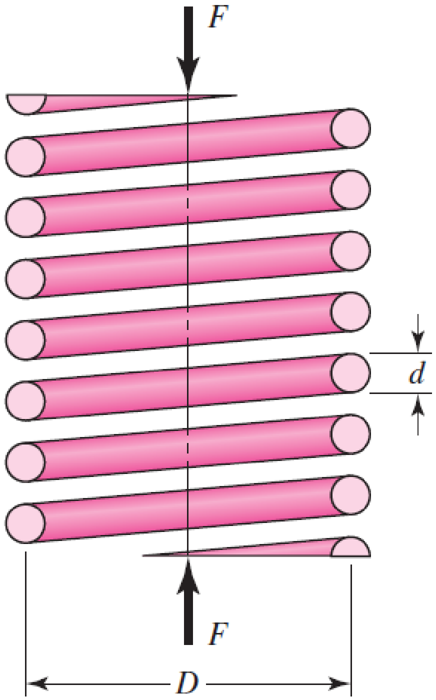
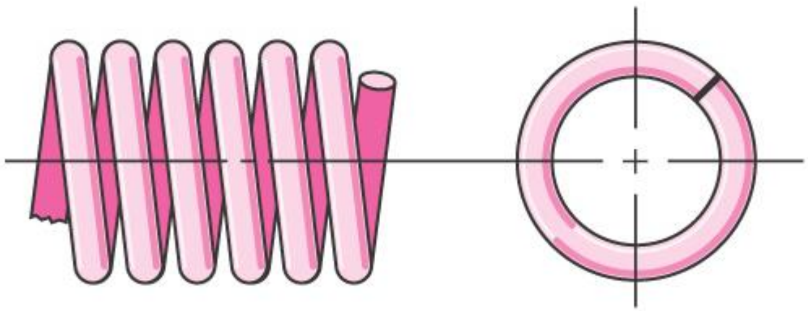
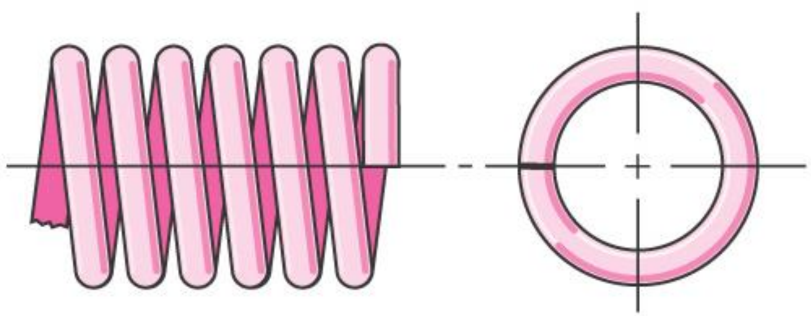


Fig. 10–1*a*

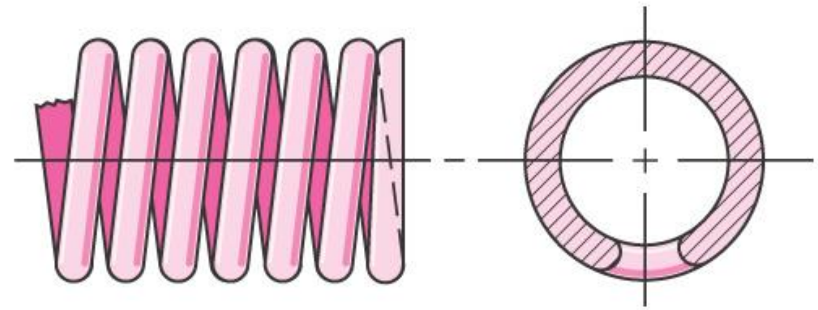
Ends of Compression Springs



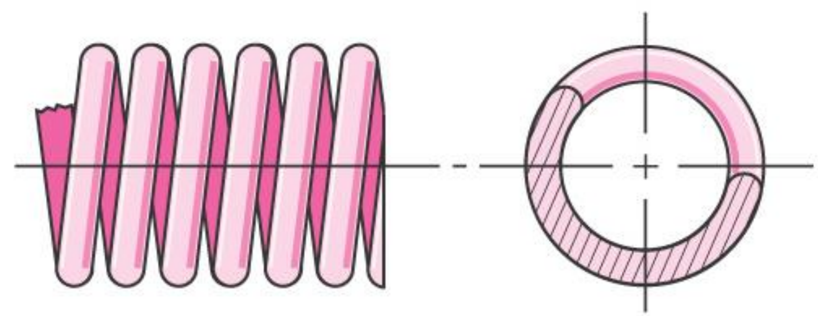
(*a*) Plain end, right hand



(*b*) Squared or closed end, right hand



(*c*) Squared and ground end, left hand



(*d*) Plain end, ground, left hand

Fig. 10–2

Formulas for Compression Springs With Different Ends

Table 10–1

Term	Type of Spring Ends: Plain	Plain and Ground	Squared or Closed	Squared and Ground
End coils, N_e	0	1	2	2
Total coils, N_t	N_a	$N_a + 1$	$N_a + 2$	$N_a + 2$
Free length, L_0	$pN_a + d$	$p(N_a + 1)$	$pN_a + 3d$	$pN_a + 2d$
Solid length, L_s	$d(N_t + 1)$	dN_t	$d(N_t + 1)$	dN_t
Pitch, p	$(L_0 - d)/N_a$	$L_0/(N_a + 1)$	$(L_0 - 3d)/N_a$	$(L_0 - 2d)/N_a$

N_a is the number of active coils

Set Removal

- *Set removal* or *presetting* is a process used in manufacturing a spring to induce useful residual stresses.
- The spring is made longer than needed, then compressed to solid height, intentionally exceeding the yield strength.
- This operation *sets* the spring to the required final free length.
- Yielding induces residual stresses opposite in direction to those induced in service.
- 10 to 30 percent of the initial free length should be removed.
- Set removal is not recommended when springs are subject to fatigue.

Critical Deflection for Stability

- Buckling type of instability can occur in compression springs when the deflection exceeds the *critical deflection* y_{cr}

$$y_{cr} = L_0 C_1' \left[1 - \left(1 - \frac{C_2'}{\lambda_{eff}^2} \right)^{1/2} \right] \qquad (10\text{–}10)$$

- L_{eff} is the *effective slenderness ratio*

$$\lambda_{eff} = \frac{\alpha L_0}{D} \qquad (10\text{–}11)$$

- α is the *end-condition constant*, defined on the next slide
- C'_1 and C'_2 are elastic constants

$$C_1' = \frac{E}{2(E-G)}$$

$$C_2' = \frac{2\pi^2(E-G)}{2G+E}$$

End-Condition Constant

- The α term in Eq. (10–11) is the *end-condition constant.*
- It accounts for the way in which the ends of the spring are supported.
- Values are given in Table 10–2.

End Condition	Constant α
Spring supported between flat parallel surfaces (fixed ends)	0.5
One end supported by flat surface perpendicular to spring axis (fixed); other end pivoted (hinged)	0.707
Both ends pivoted (hinged)	1
One end clamped; other end free	2

*Ends supported by flat surfaces must be squared and ground.

Table 10–2

Absolute Stability

- Absolute stability occurs when, in Eq. (10–10),

$$C_2' / \lambda_{\text{eff}}^2 > 1$$

- This results in the condition for absolute stability

$$L_0 < \frac{\pi D}{\alpha}\left[\frac{2(E-G)}{2G+E}\right]^{1/2} \qquad (10\text{–}12)$$

- For steels, this turns out to be

$$L_0 < 2.63\frac{D}{\alpha} \qquad (10\text{–}13)$$

Some Common Spring Steels

- Hard-drawn wire (0.60-0.70*C*)
  - Cheapest general-purpose
  - Use only where life, accuracy, and deflection are not too important
- Oil-tempered wire (0.60-0.70*C*)
  - General-purpose
  - Heat treated for greater strength and uniformity of properties
  - Often used for larger diameter spring wire
- Music wire (0.80-0.95*C*)
  - Higher carbon for higher strength
  - Best, toughest, and most widely used for small springs
  - Good for fatigue

Some Common Spring Steels

- Chrome-vanadium
  - Popular alloy spring steel
  - Higher strengths than plain carbon steels
  - Good for fatigue, shock, and impact
- Chrome-silicon
  - Good for high stresses, long fatigue life, and shock

Strength of Spring Materials

- With small wire diameters, strength is a function of diameter.
- A graph of tensile strength vs. wire diameter is almost a straight line on log-log scale.
- The equation of this line is

$$S_{ut} = \frac{A}{d^m} \qquad (10\text{–}14)$$

where A is the intercept and m is the slope.

- Values of A and m for common spring steels are given in Table 10–4.

Constants for Estimating Tensile Strength

$$S_{ut} = \frac{A}{d^m} \qquad (10\text{–}14)$$

Material	ASTM No.	Exponent m	Diameter, in	A, kpsi · in^m	Diameter, mm	A, MPa · mm^m	Relative Cost of Wire
Music wire*	A228	0.145	0.004–0.256	201	0.10–6.5	2211	2.6
OQ&T wire†	A229	0.187	0.020–0.500	147	0.5–12.7	1855	1.3
Hard-drawn wire‡	A227	0.190	0.028–0.500	140	0.7–12.7	1783	1.0
Chrome-vanadium wire§	A232	0.168	0.032–0.437	169	0.8–11.1	2005	3.1
Chrome-silicon wire‖	A401	0.108	0.063–0.375	202	1.6–9.5	1974	4.0
302 Stainless wire#	A313	0.146	0.013–0.10	169	0.3–2.5	1867	7.6–11
		0.263	0.10–0.20	128	2.5–5	2065	
		0.478	0.20–0.40	90	5–10	2911	
Phosphor-bronze wire**	B159	0	0.004–0.022	145	0.1–0.6	1000	8.0
		0.028	0.022–0.075	121	0.6–2	913	
		0.064	0.075–0.30	110	2–7.5	932	

Table 10–4

Estimating Torsional Yield Strength

- Since helical springs experience shear stress, shear yield strength is needed.
- If actual data is not available, estimate from tensile strength
- Assume yield strength is between 60-90% of tensile strength

$$0.6S_{ut} \leq S_{sy} \leq 0.9S_{ut}$$

- Assume the distortion energy theory can be employed to relate the shear strength to the normal strength.

$$S_{sy} = 0.577S_y$$

- This results in

$$0.35S_{ut} \leq S_{sy} \leq 0.52S_{ut} \tag{10–15}$$

Mechanical Properties of Some Spring Wires (Table 10–5)

Material	Elastic Limit, Percent of S_{ut} Tension	Elastic Limit, Percent of S_{ut} Torsion	Diameter d, in	E Mpsi	E GPa	G Mpsi	G GPa
Music wire A228	65–75	45–60	<0.032	29.5	203.4	12.0	82.7
			0.033–0.063	29.0	200	11.85	81.7
			0.064–0.125	28.5	196.5	11.75	81.0
			>0.125	28.0	193	11.6	80.0
HD spring A227	60–70	45–55	<0.032	28.8	198.6	11.7	80.7
			0.033–0.063	28.7	197.9	11.6	80.0
			0.064–0.125	28.6	197.2	11.5	79.3
			>0.125	28.5	196.5	11.4	78.6
Oil tempered A239	85–90	45–50		28.5	196.5	11.2	77.2
Valve spring A230	85–90	50–60		29.5	203.4	11.2	77.2
Chrome-vanadium A231	88–93	65–75		29.5	203.4	11.2	77.2
A232	88–93			29.5	203.4	11.2	77.2
Chrome-silicon A401	85–93	65–75		29.5	203.4	11.2	77.2
Stainless steel							
A313*	65–75	45–55		28	193	10	69.0
17-7PH	75–80	55–60		29.5	208.4	11	75.8
414	65–70	42–55		29	200	11.2	77.2
420	65–75	45–55		29	200	11.2	77.2
431	72–76	50–55		30	206	11.5	79.3
Phosphor-bronze B159	75–80	45–50		15	103.4	6	41.4
Beryllium-copper B197	70	50		17	117.2	6.5	44.8
	75	50–55		19	131	7.3	50.3
Inconel alloy X-750	65–70	40–45		31	213.7	11.2	77.2

Maximum Allowable Torsional Stresses

Table 10–6

Maximum Allowable Torsional Stresses for Helical Compression Springs in Static Applications

Source: Robert E. Joerres, "Springs," Chap. 6 in Joseph E. Shigley, Charles R. Mischke, and Thomas H. Brown, Jr. (eds.), *Standard Handbook of Machine Design*, 3rd ed., McGraw-Hill, New York, 2004.

	Maximum Percent of Tensile Strength	
Material	**Before Set Removed (includes K_W or K_B)**	**After Set Removed (includes K_s)**
Music wire and cold-drawn carbon steel	45	60–70
Hardened and tempered carbon and low-alloy steel	50	65–75
Austenitic stainless steels	35	55–65
Nonferrous alloys	35	55–65

Critical Frequency of Helical Springs

- When one end of a spring is displaced rapidly, a wave called a *spring surge* travels down the spring.
- If the other end is fixed, the wave can reflect back.
- If the wave frequency is near the natural frequency of the spring, resonance may occur resulting in extremely high stresses.
- Catastrophic failure may occur, as shown in this valve-spring from an over-revved engine.

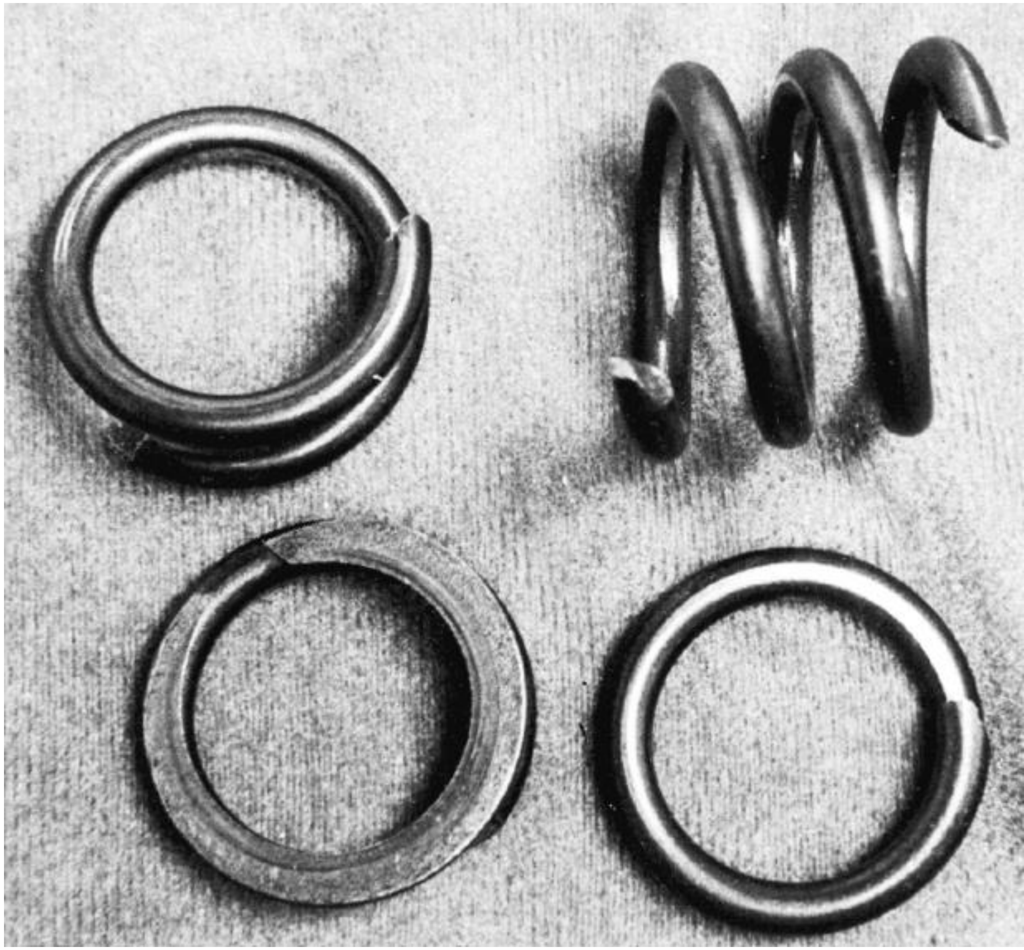


Fig. 10–4

Critical Frequency of Helical Springs

- The governing equation is the wave equation

$$\frac{\partial^2 u}{\partial x^2} = \frac{W}{kgl^2}\frac{\partial^2 u}{\partial t^2} \tag{10-24}$$

where k = spring rate

g = acceleration due to gravity

l = length of spring

W = weight of spring

x = coordinate along length of spring

u = motion of any particle at distance x

Critical Frequency of Helical Springs

- The solution to this equation is harmonic and depends on the given physical properties as well as the end conditions.
- The harmonic, natural, frequencies for a spring placed between two flat and parallel plates, in radians per second, are

$$\omega = m\pi\sqrt{\frac{kg}{W}} \qquad m = 1, 2, 3, \ldots$$

- In cycles per second, or hertz,

$$f = \frac{1}{2}\sqrt{\frac{kg}{W}} \tag{10–25}$$

- With one end against a flat plate and the other end free,

$$f = \frac{1}{4}\sqrt{\frac{kg}{W}} \tag{10–26}$$

Critical Frequency of Helical Springs

- The weight of a helical spring is

$$W = AL\gamma = \frac{\pi d^2}{4}(\pi D N_a)(\gamma) = \frac{\pi^2 d^2 D N_a \gamma}{4} \qquad (10\text{–}27)$$

- The fundamental critical frequency should be greater than 15 to 20 times the frequency of the force or motion of the spring.
- If necessary, redesign the spring to increase k or decrease W.

Fatigue Loading of Helical Compression Springs

- Zimmerli found that size, material, and tensile strength have no effect on the endurance limits of spring steels in sizes under 3/8 in (10 mm).
- Testing found the endurance strength components for infinite life to be

Unpeened:

$$S_{sa} = 35 \text{ kpsi } (241 \text{ MPa}) \qquad S_{sm} = 55 \text{ kpsi } (379 \text{ MPa}) \tag{10–28}$$

Peened:

$$S_{sa} = 57.5 \text{ kpsi } (398 \text{ MPa}) \qquad S_{sm} = 77.5 \text{ kpsi } (534 \text{ MPa}) \tag{10–29}$$

- These constant values are used with Gerber or Goodman failure criteria to find the endurance limit.

Fatigue Loading of Helical Compression Springs

- For example, with an unpeened spring with S_{su} = 211.5 kpsi, the Gerber ordinate intercept for shear, from Eq. (6-42), is

$$S_{se} = \frac{S_{sa}}{1 - \left(\dfrac{S_{sm}}{S_{su}}\right)^2} = \frac{35}{1 - \left(\dfrac{55}{211.5}\right)^2} = 37.5 \text{ kpsi}$$

- For the Goodman criterion, it would be S_{se} = 47.3 kpsi.
- Each possible wire size would change the endurance limit since S_{su} is a function of wire size.

Fatigue Loading of Helical Compression Springs

- It has been found that for polished, notch-free, cylindrical specimens subjected to torsional shear stress, the maximum alternating stress that may be imposed is constant and independent of the mean stress.
- Many compression springs approach these conditions.
- This failure criterion is known as the *Sines failure criterion*.

Torsional Modulus of Rupture

- The torsional modulus of rupture S_{su} will be needed for the fatigue diagram.
- Lacking test data, the recommended value is

$$S_{su} = 0.67 S_{ut} \tag{10–30}$$

Stresses for Fatigue Loading

- From the standard approach, the alternating and midrange forces are

$$F_a = \frac{F_{\max} - F_{\min}}{2} \qquad (10\text{–}31a)$$

$$F_m = \frac{F_{\max} + F_{\min}}{2} \qquad (10\text{–}31b)$$

- The alternating and midrange stresses are

$$\tau_a = K_B \frac{8 F_a D}{\pi d^3} \qquad (10\text{–}32)$$

$$\tau_m = K_B \frac{8 F_m D}{\pi d^3} \qquad (10\text{–}33)$$

Extension Springs

- Extension springs are similar to compression springs within the body of the spring.
- To apply tensile loads, hooks are needed at the ends of the springs.
- Some common hook types:

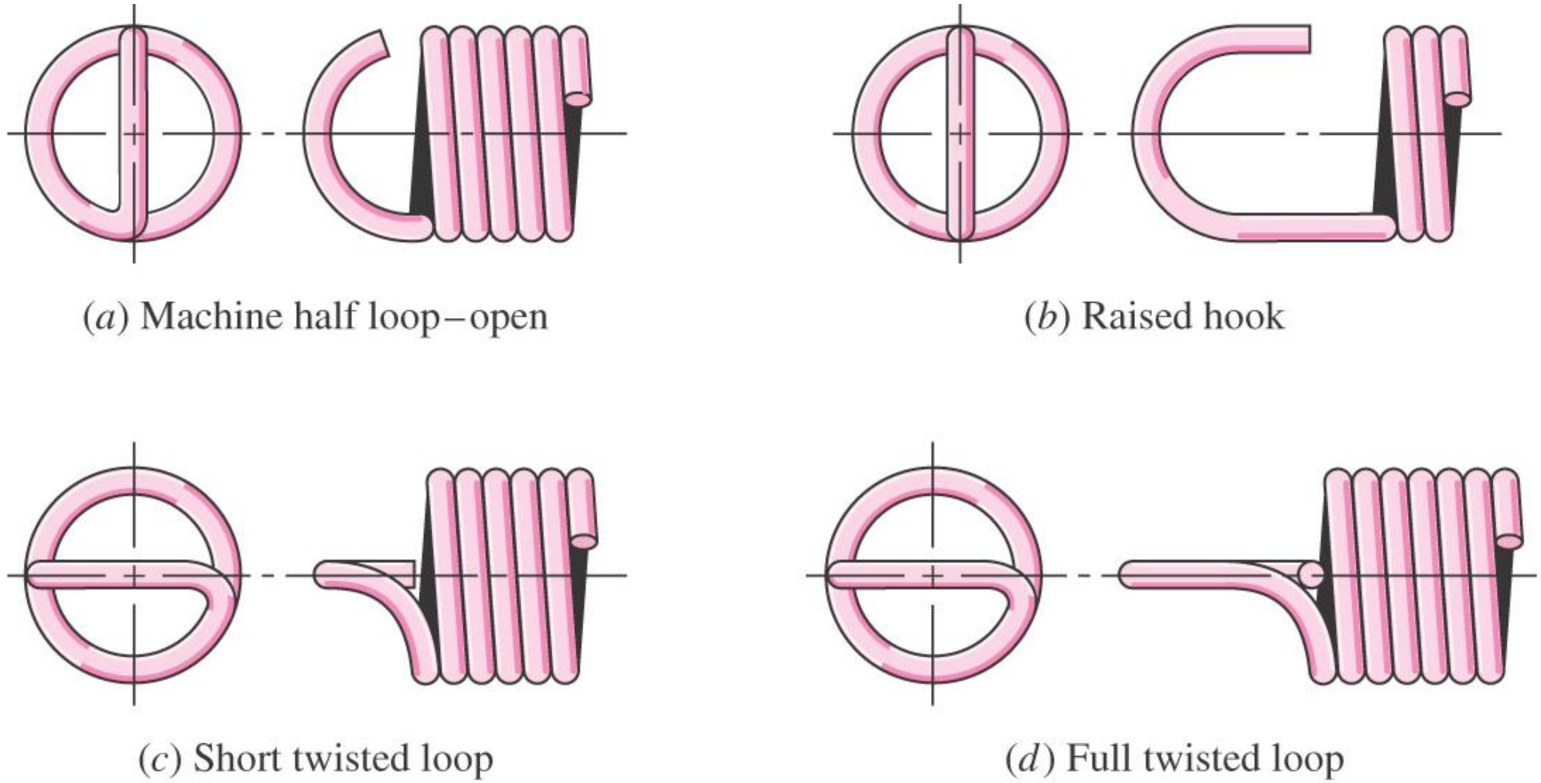


Fig. 10–5

Stress in the Hook

- In a typical hook, a critical stress location is at point *A*, where there is bending and axial loading.

$$\sigma_A = F\left[(K)_A \frac{16D}{\pi d^3} + \frac{4}{\pi d^2}\right] \qquad (10\text{–}34)$$

- $(K)_A$ is a bending stress-correction factor for curvature

$$(K)_A = \frac{4C_1^2 - C_1 - 1}{4C_1(C_1 - 1)} \qquad C_1 = \frac{2r_1}{d} \qquad (10\text{–}35)$$

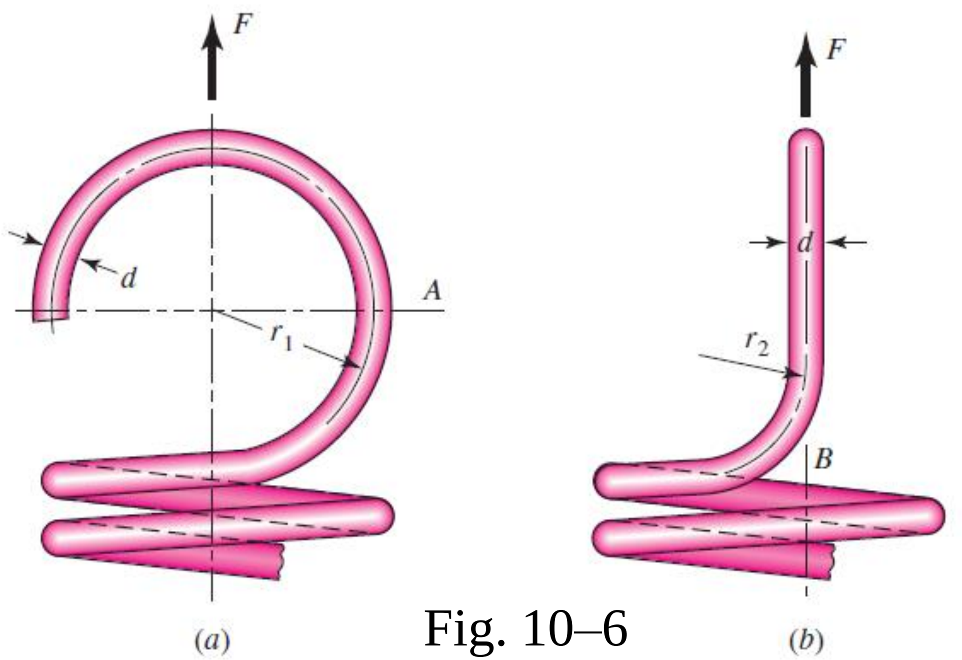


Fig. 10–6

Stress in the Hook

- Another potentially critical stress location is at point *B*, where there is primarily torsion.

$$\tau_B = (K)_B \frac{8FD}{\pi d^3} \tag{10–36}$$

- $(K)_B$ is a stress-correction factor for curvature.

$$(K)_B = \frac{4C_2 - 1}{4C_2 - 4} \qquad C_2 = \frac{2r_2}{d} \tag{10–37}$$

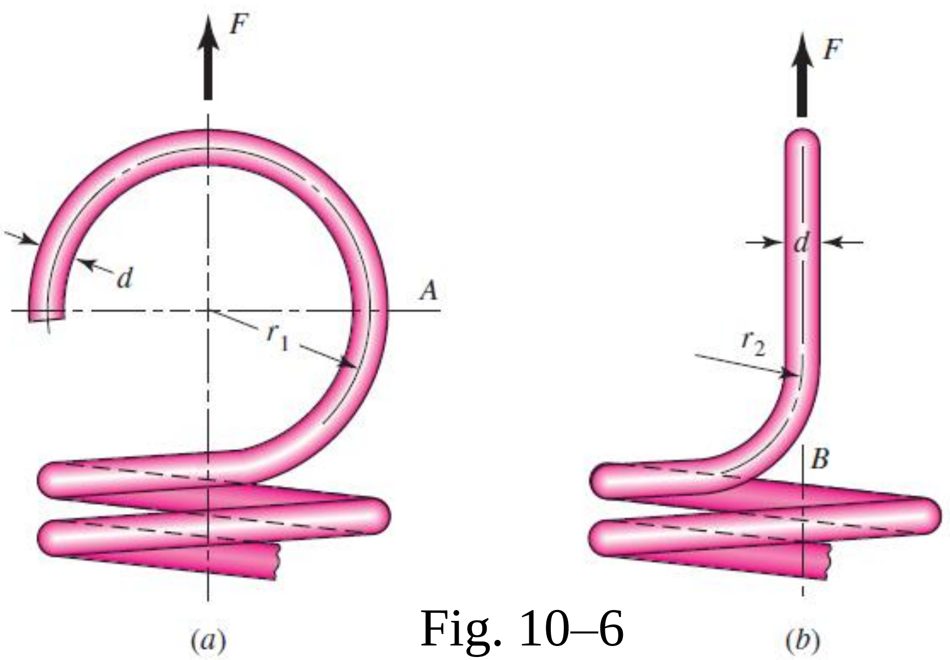


Fig. 10–6

An Alternate Hook Design

- This hook design reduces the coil diameter at point *A*.

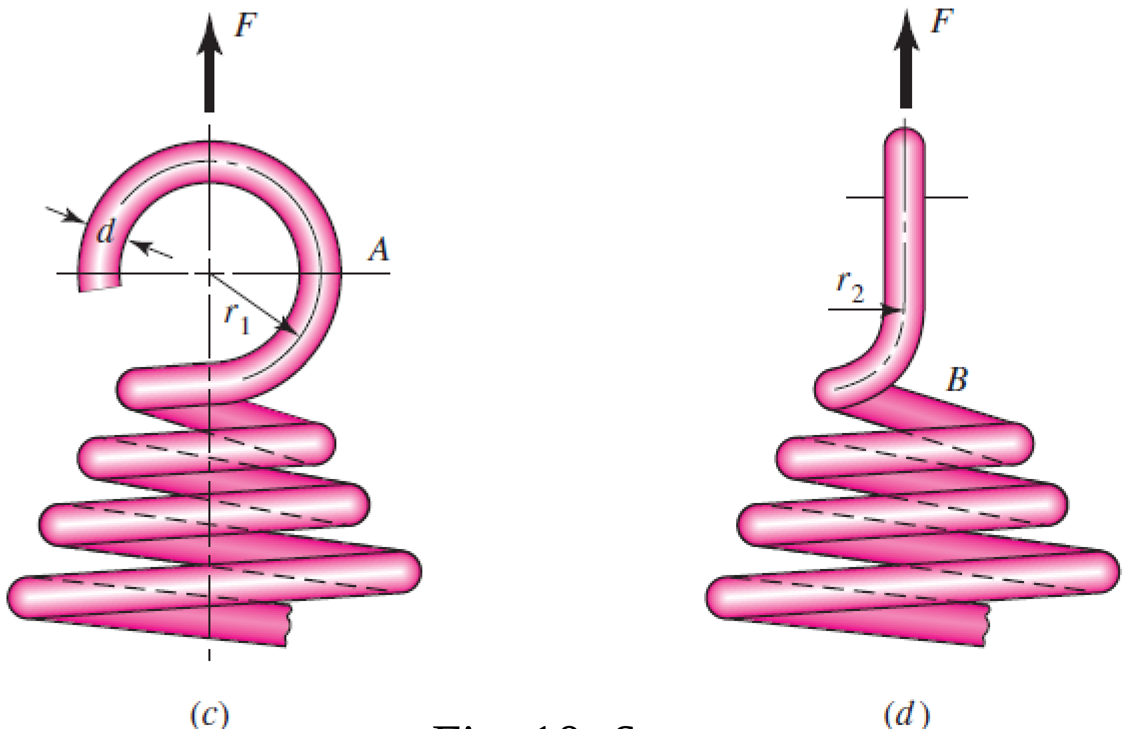


Fig. 10–6

Close-wound Extension Springs

- Extension springs are often made with coils in contact with one another, called *close-wound*.
- Including some initial tension in close-wound springs helps hold the free length more accurately.
- The load-deflection curve is offset by this initial tension F_i

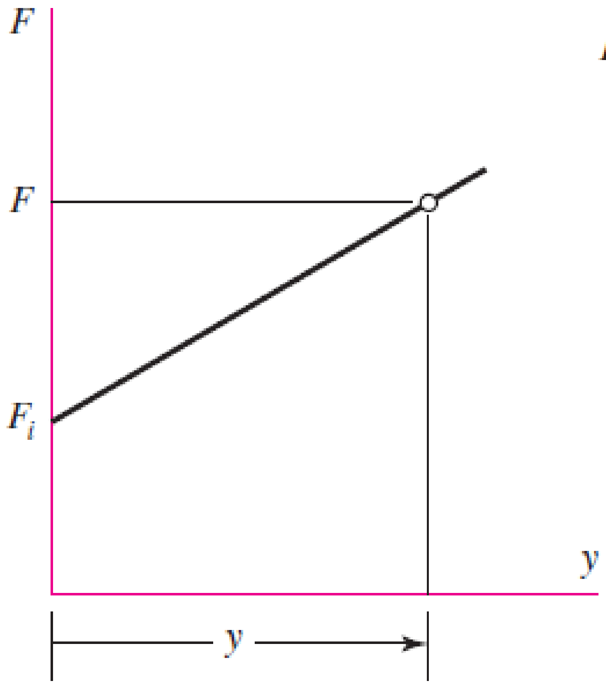


Fig. 10–7 (a)

$$F = F_i + ky \tag{10–38}$$

Terminology of Extension Spring Dimensions

- The free length is measured inside the end hooks.

$$L_0 = 2(D - d) + (N_b + 1)d = (2C - 1 + N_b)d \qquad (10\text{–}39)$$

- The hooks contribute to the spring rate. This can be handled by obtaining an equivalent number of active coils.

$$N_a = N_b + \frac{G}{E} \qquad (10\text{–}40)$$

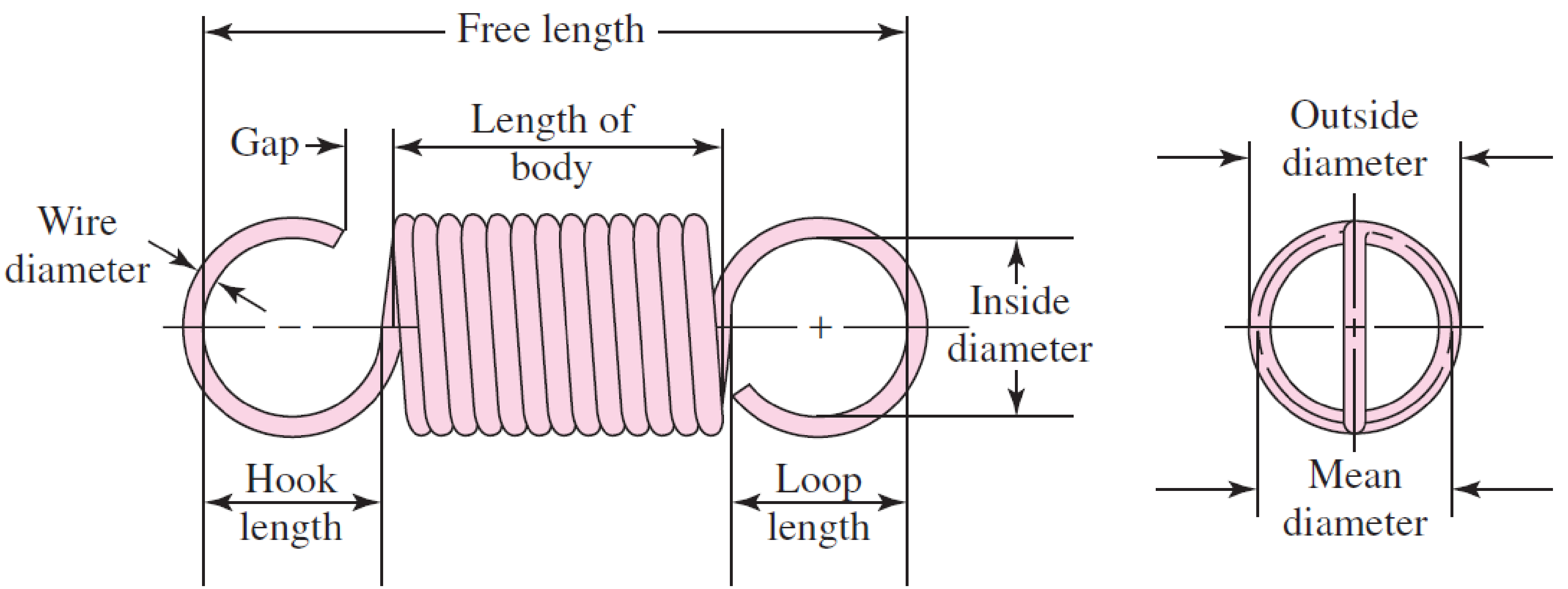


Fig. 10–7 (*b*)

Initial Tension in Close-Wound Springs

- Initial tension is created by twisting the wire as it is wound onto a mandrel.
- When removed from the mandrel, the initial tension is locked in because the spring cannot get any shorter.
- The amount of initial tension that can routinely be incorporated is shown.
- The two curves bounding the preferred range is given by

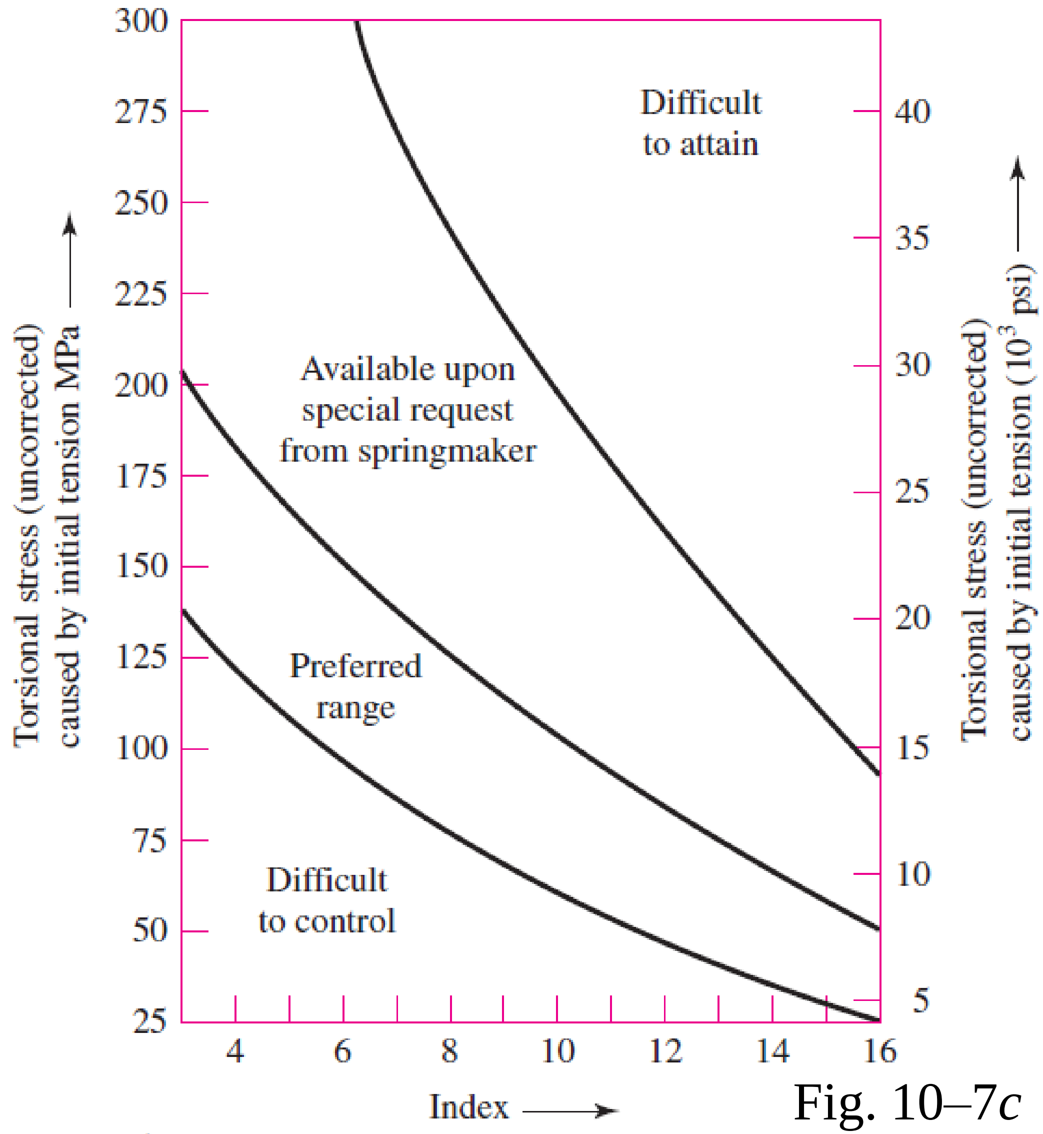


Fig. 10–7*c*

$$\tau_i = \frac{33\,500}{\exp(0.105C)} \pm 1000\left(4 - \frac{C-3}{6.5}\right) \text{ psi} \qquad (10\text{–}41)$$

Guidelines for Maximum Allowable Stresses

- Recommended maximum allowable stresses, corrected for curvature effect, for static applications is given in Table 10–7.

Table 10–7

Materials	Percent of Tensile Strength: In Torsion, Body	In Torsion, End	In Bending, End
Patented, cold-drawn or hardened and tempered carbon and low-alloy steels	45–50	40	75
Austenitic stainless steel and nonferrous alloys	35	30	55

This information is based on the following conditions: set not removed and low temperature heat treatment applied. For springs that require high initial tension, use the same percent of tensile strength as for end.

Belleville Springs

- The *Belleville spring* is a coned-disk spring with unique properties
- It has a non-linear spring constant
- With $h/t \geq 2.83$, the S curve can be useful for snap-acting mechanisms
- For $1.41 \leq h/t \leq 2.1$ the flat central portion provides constant load for a considerable deflection range

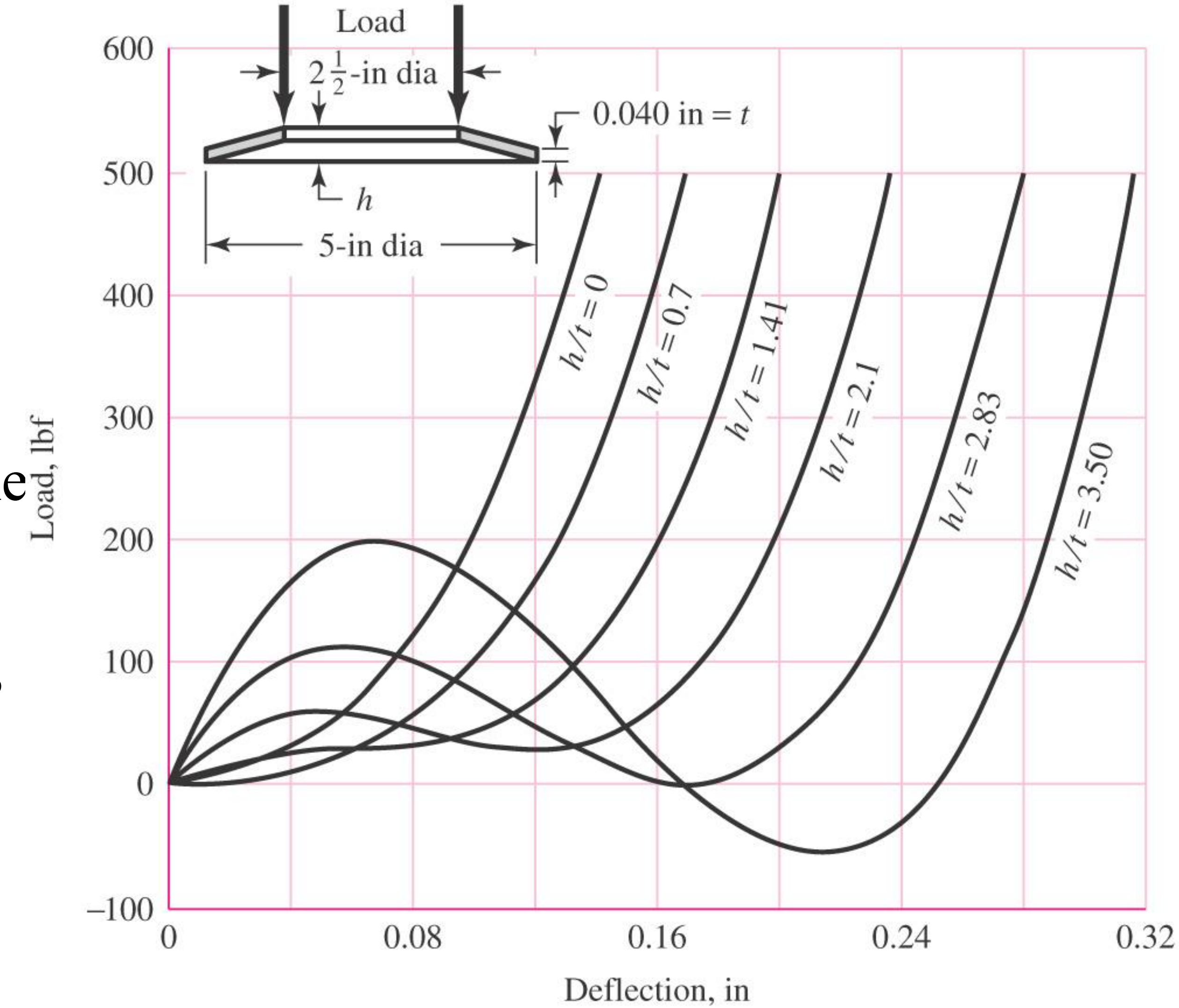


Fig. 10–11

Constant-Force Springs

- The extension spring shown is made of slightly curved strip steel, not flat.
- The fore required to uncoil it remains constant.
- Known as a *constant-force spring*.

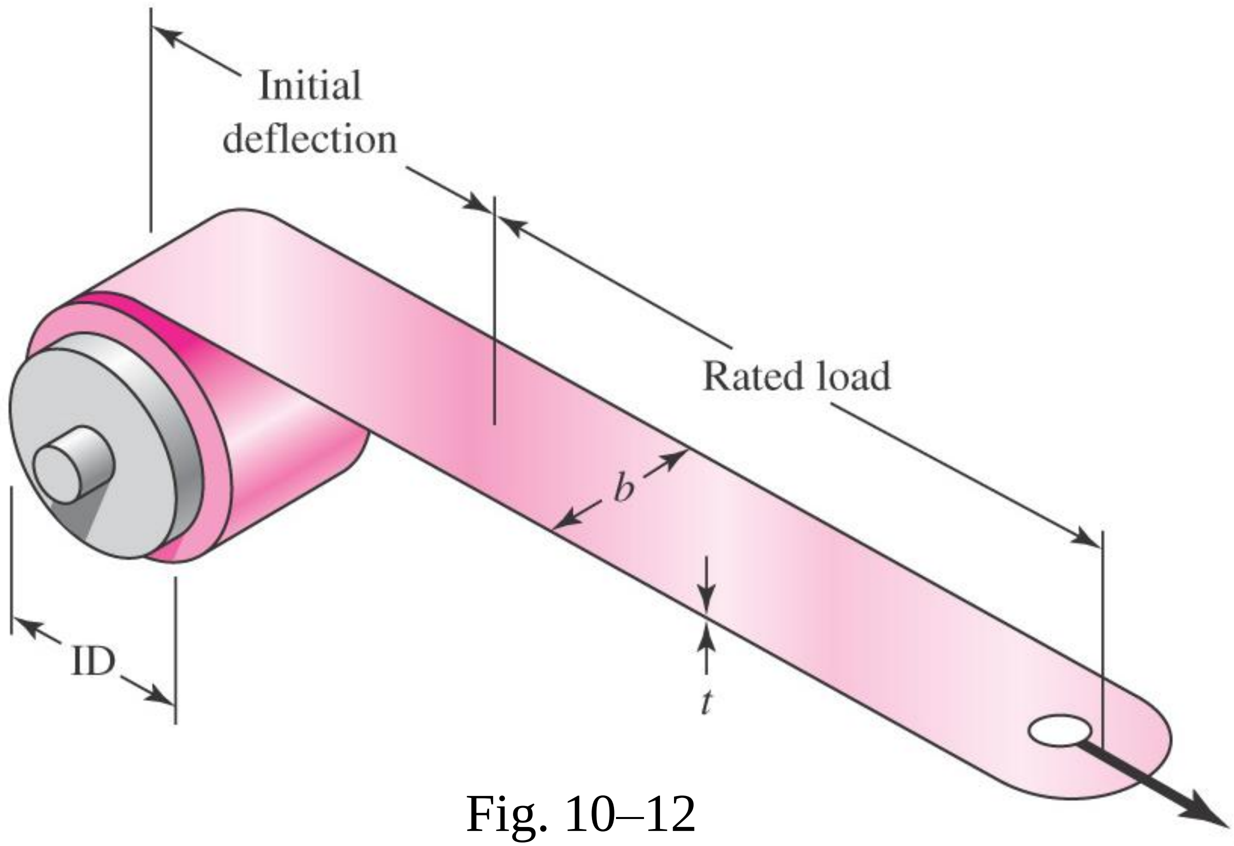
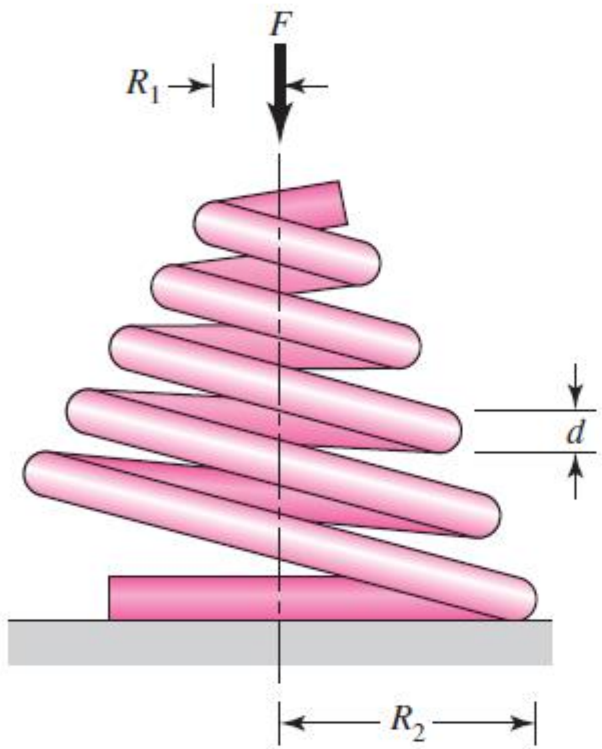


Fig. 10–12

Conical Spring

- A *conical spring* is wound in the shape of a cone.
- Most are compression springs, made with round wire.
- The principal advantage is that the solid height is only a single wire diameter.



Volute Spring

- A *volute spring* is a conical spring made from a wide, thin strip, or "flat", of material wound on the flat so that the coils fit inside one another.
- Since the coils do not stack on each other, the solid height is the width of the strip.
- A variable-spring scale is obtained by permitting the coils to contact the support.
- As deflection increases (in compression), the number of active coils decreases.

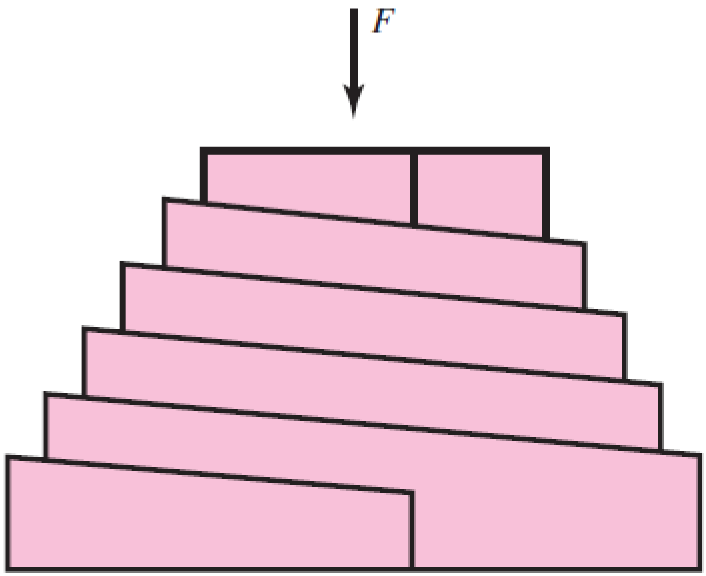


Fig. 10–13*a*

Constant-Stress Cantilever Spring

- A uniform-section cantilever spring made from flat stock has stress which is proportional to the distance *x*.

$$\sigma = \frac{M}{I/c} = \frac{Fx}{I/c} \qquad (a)$$

- It is often economical to proportion the width *b* to obtain uniform stress, independent of *x*.

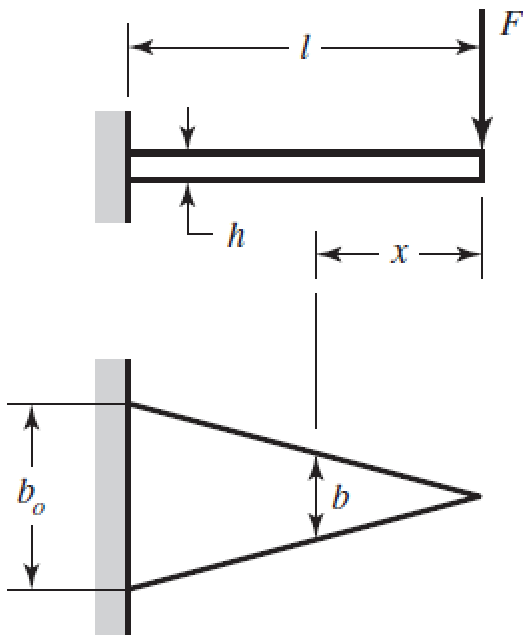


Fig. 10–13*b*

Constant-Stress Cantilever Spring

- For a rectangular section, $I/c = bh^2/6$.
- Combining with Eq. (*a*),

$$\frac{bh^2}{6} = \frac{Fx}{\sigma}$$

- Solving for *b*,

$$b = \frac{6Fx}{h^2\sigma}$$

- Since *b* is linearly related to *x*, the width b_o at the base is

$$b_o = \frac{6Fl}{h^2\sigma} \qquad (10\text{–}62)$$

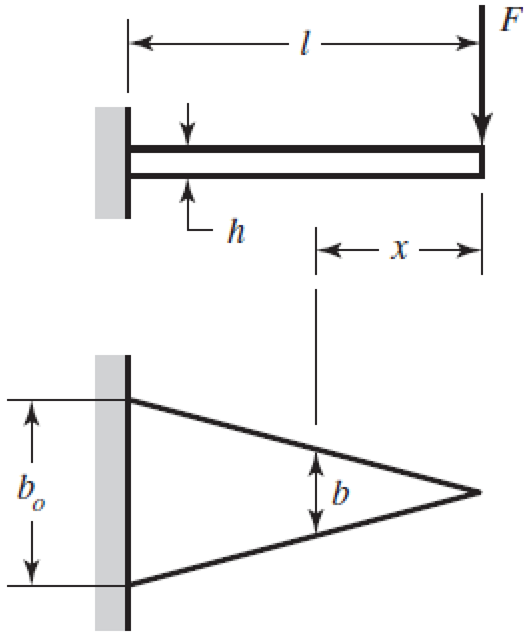


Fig. 10–13*b*

Constant-Stress Cantilever Spring

- Apply Castigliano's method to obtain deflection and spring constant equations.
- The width is a function of *x*,

$$b = b_o x / l$$

- Integrating Castigliano's deflection equation with *M* and *I* both functions of *x*,

$$y = \int_0^l \frac{M(\partial M/\partial F)}{EI}\,dx = \frac{1}{E}\int_0^l \frac{-Fx(-x)}{\frac{1}{12}(b_o x/l)h^3}\,dx$$

$$= \frac{12Fl}{b_o h^3 E}\int_0^l x\,dx = \frac{6Fl^3}{b_o h^3 E} \qquad (10\text{–}63)$$

- Thus, the spring constant, $k = F/y$, is

$$k = \frac{b_o h^3 E}{6l^3} \qquad (10\text{–}64)$$

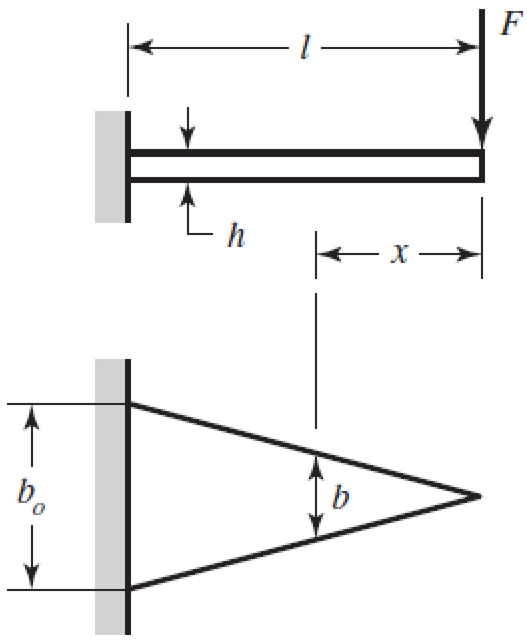


Fig. 10–13*b*