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Excitation response relations for linear systems

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Introduction

- We turn now to the response characteristics of physical systems. Before considering what happens when a system is subjected to random excitation, we must deal first with various different methods for describing the response of a general system to deterministic (non-random) excitation.
- The general system may be a vibrating structure or machine, or a complete building, or a small electric circuit. Whatever it is, we assume that there are a number of inputs $x_1(t)$,$x_2(t)$,$x_3(t)$ etc., which constitute the excitation and a number of outputs $y_1(t)$,$y_2(t)$,$y_3(t)$ etc., which constitute the response.
- The x(t) and y(t) may be forces, pressures, displacements, velocities, accelerations, voltages, currents, etc., or a mixture of all these.

Introduction

- We shall restrict ourselves to linear systems for which each response variable y(t) is related to the excitation by a linear differential equation of the form:

$$a_n \frac{d^n y_1}{dt^n} + a_{n-1} \frac{d^{n-1} y_1}{dt^{n-1}} + \cdots + a_1 \frac{dy_1}{dt} + a_0 y_1 =$$

$$= \left\{ b_r \frac{d^r x_1}{dt^r} + b_{r-1} \frac{d^{r-1} x_1}{dt^{r-1}} + \cdots + b_1 \frac{dx_1}{dt} + b_0 x_1 + \right.$$

$$+ c_s \frac{d^s x_2}{dt^s} + c_{s-1} \frac{d^{s-1} x_2}{dt^{s-1}} + \cdots + c_1 \frac{dx_2}{dt} + c_0 x_2 +$$

$$+ d_t \frac{d^t x_3}{dt^t} + d_{t-1} \frac{d^{t-1} x_3}{dt^{t-1}} + \cdots + d_1 \frac{dx_3}{dt} + d_0 x_3 +$$

$$\left. + \ldots \ldots \ldots \ldots \ldots \ldots \right\}.$$

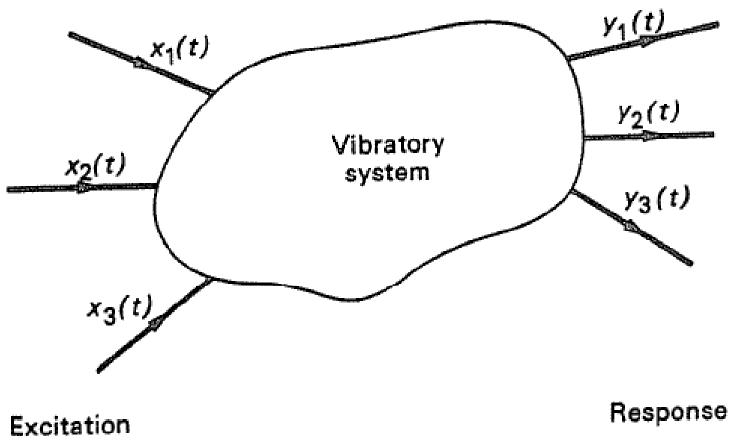
Introduction

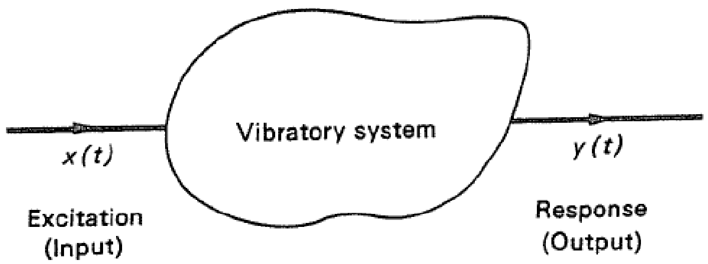
- These equations are linear and the principle of superposition applies. The coefficients a,b,c, and d may in general be functions of time, but we shall consider only cases when they are constant which means that the vibrating system does not change its characteristic with time.

- On account of the principle of superposition, our problem is greatly simplified because we can consider how each output variable responds to a single input variable alone, and then just add together the separate responses to many input variables in order to obtain the response to the combined excitation at many points. Of course, the assumption of linearity is a bold one, but since vibrations usually only involve small displacements from equilibrium, it is very often not too far from the truth.

Introduction

- We can therefore simplify the system we should consider on the left to that shown in the figure on the right.





Classical approach

- If the equations of motion for the constant parameter linear system can be determined, then there is a known linear differential equation relating y(t) and x(t) with the form:

$$a_n \frac{d^n y}{dt^n} + a_{n-1} \frac{d^{n-1} y}{dt^{n-1}} + \cdots + a_1 \frac{dy}{dt} + a_0 y =$$

$$= b_r \frac{d^r x}{dt^r} + b_{r-1} \frac{d^{r-1} x}{dt^{r-1}} + \cdots + b_1 \frac{dx}{dt} + b_0 x.$$

- For given excitation x(t) and given initial conditions, this equation can be solved by classical methods to give a complete solution for y(t). However, such an approach is not usually helpful for random vibration problems for two reasons. First the above differential equation is seldom obtainable directly because there is inadequate data available and simple experimental methods for finding the coefficients a and b are not available. Secondly, even if the differential equation is known, a complete time history for y(t) can only be calculated if we have a complete time history x(t), and for random vibration problems, this data is not of course available. In order to calculate average values of the output variables, it is more convenient to concentrate on alternative ways of representing the relationship between y(t) and x(t).

Frequency response method

- A completely different method of describing the dynamic characteristics of a linear system is to determine the response to a sine wave input. If the input is a constant amplitude sine wave of fixed frequency

$$x(t) = x_0 \sin \omega t$$

- Then from

$$a_n \frac{d^n y}{dt^n} + a_{n-1} \frac{d^{n-1} y}{dt^{n-1}} + \cdots + a_1 \frac{dy}{dt} + a_0 y =$$

$$= b_r \frac{d^r x}{dt^r} + b_{r-1} \frac{d^{r-1} x}{dt^{r-1}} + \cdots + b_1 \frac{dx}{dt} + b_0 x.$$

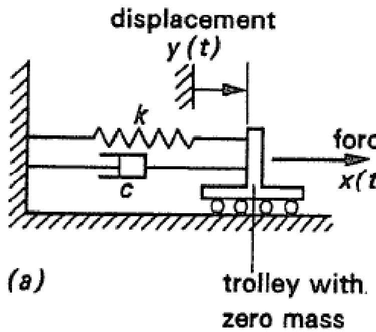
- The steady state output must also be a sine wave of fixed amplitude, the same frequency ω and phase difference ϕ so that

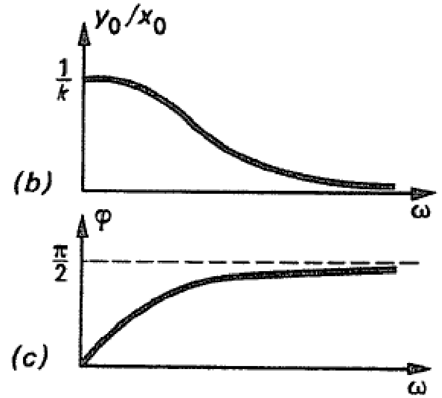
$$y(t) = y_0 \sin(\omega t - \phi).$$

Frequency response method

- Information about the amplitude ratio y_o/x_o and the phase angle ϕ defines the transmission characteristics or transfer function of the system at the fixed frequency ω. By making measurements at a series of closely spaced frequencies, amplitude ratio and phase angle can be plotted as a function of frequency, and in theory if the frequency range extends from zero to infinity then the dynamic characteristics of the system are completely defined.

  Example: Determine the amplitude ratio and phase angle for the transmission of sine wave excitation through the spring-damper system of the figure. The excitation is the force x(t) and the response is the displacement y(t).





Frequency response method

- For a linear spring of stiffness k and a linear viscous damper of coefficient c, the equation of motion is

$$c\dot{y} + ky = x(t)$$

and so when x(t) is a constant amplitude sine wave

$$x(t) = x_0 \sin \omega t$$

and y(t) is the response given by:

$$y(t) = y_0 \sin(\omega t - \phi).$$

then:

$$c y_0 \omega \cos(\omega t - \phi) + k y_0 \sin(\omega t - \phi) = x_0 \sin \omega t$$

which gives collecting terms:

$$y_0 \sin \omega t \left\{ c\omega \sin \phi + k \cos \phi - \frac{x_0}{y_0} \right\} + y_0 \cos \omega t \left\{ c\omega \cos \phi - k \sin \phi \right\} = 0.$$

For this to be true, the terms in brackets must be separately zero, so that the amplitude ratio:

Frequency response method

$$\frac{y_0}{x_0} = \frac{1}{\sqrt{(c^2\omega^2 + k^2)}}$$

and the phase angle:

$$\phi = \tan^{-1}\frac{c\omega}{k}$$

- Instead of thinking of amplitude ratio and phase angle as two separate quantities, it has become customary in vibration theory to use a single complex number to represent both quantities. This is called, the (complex) frequency response function H(ω) which is defined so that its magnitude is equal to the amplitude ratio and the ratio of its imaginary part to its real part is equal to the tangent of the phase angle. If

$$H(\omega) = A(\omega) - iB(\omega)$$

where A(ω) and B(ω) are real functions of ω, then

$$|H(\omega)| = \sqrt{(A^2 + B^2)} = \frac{y_0}{x_0} \qquad \frac{\text{Imaginary part}}{\text{Real part}} = \frac{B}{A} = \tan\phi.$$

Frequency response method

- Using complex exponential notation, we can now write using

$$e^{i\theta} = \cos\theta + i\sin\theta$$

that if the input is a sine wave of amplitude x_o

$$x(t) = x_0 \sin\omega t = x_0\{\text{the imaginary part of } e^{i\omega t}\} = x_0 \operatorname{Im}(e^{i\omega t}),$$

then the corresponding harmonic output will be: $y(t) = x_0 \operatorname{Im}\{H(\omega)e^{i\omega t}\}.$

- The proof that this is correct is easily obtained by substituting

$$H(\omega) = A(\omega) - iB(\omega)$$

into the above equation for y(t) to give after using the eqns on the rhs:

$$\begin{aligned} y(t) &= x_0 \operatorname{Im}\{A(\omega) - iB(\omega)\}(\cos\omega t + i\sin\omega t) \\ &= x_0\{A(\omega)\sin\omega t - B(\omega)\cos\omega t\} \\ &= x_0\sqrt{(A^2 + B^2)}\sin\left(\omega t - \tan^{-1}\frac{B}{A}\right) \\ &= y_0 \sin(\omega t - \phi) \end{aligned}$$

$$|H(\omega)| = \sqrt{(A^2 + B^2)} = \frac{y_0}{x_0}$$

$$\frac{\text{Imaginary part}}{\text{Real part}} = \frac{B}{A} = \tan\phi.$$

Frequency response method

- In summary, therefore, if a constant amplitude harmonic input given by:

$$x(t) = x_0 \, e^{i\omega t}$$

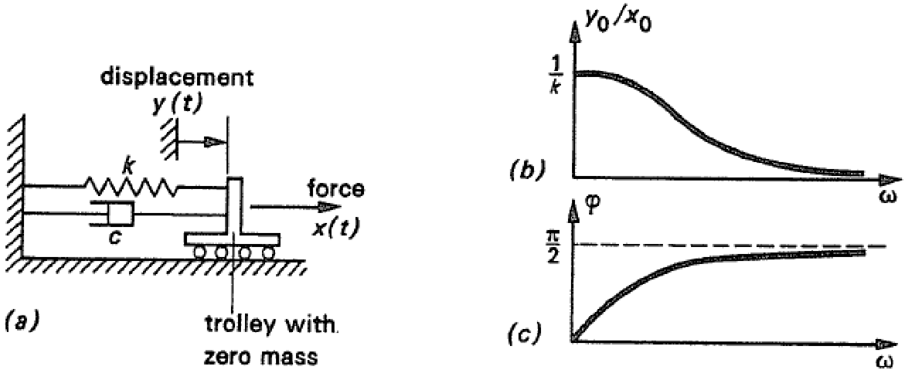
is applied to a linear system, the corresponding output y(t) will be given by:

$$y(t) = H(\omega) x_0 \, e^{i\omega t}$$

where H(ω) is the system's complex frequency response function evaluated at angular frequency ω. In interpreting the above two equations, we mean either the "real part" or "the imaginary part" of the right hand side of the equations (but not both) according to the convention agreed on. The magnitude of H(ω) gives the ratio of the amplitudes of y(t) and x(t) and its argument gives the phase angle between y(t) and x(t). Notice that the quantity x_o in the above equations need not necessarily be real; its magnitude gives the amplitude of x(t) while the magnitude of $H(\omega)x_o$ gives the amplitude of y(t).

Example

- Calculate the frequency response function for the system of the figure.



- The equation of motion is $c\dot{y} + ky = x.$
- Putting $x = x_0 \mathrm{e}^{i\omega t}$ (where either "the imaginary part of" or "the real part of" is understood) and

$$y = H(\omega) x_0 \mathrm{e}^{i\omega t}$$

gives

$$(ci\omega + k) H(\omega) \mathrm{e}^{i\omega t} = \mathrm{e}^{i\omega t}$$

or

$$H(\omega) = \frac{1}{k + ic\omega} = A(\omega) - iB(\omega).$$

Example

- The amplitude ratio

$$\frac{y_0}{x_0} = |H(\omega)| = \frac{1}{\sqrt{(k^2 + c^2\omega^2)}}$$

and the phase angle is given by:

$$\tan\phi = \frac{B(\omega)}{A(\omega)} = \frac{c\omega}{k}$$

which checks the previous example. Notice that H(ω) has dimensions (in this case those of displacement/force) and that the output lags behind the input by angle ϕ.

Impulse response method

- The frequency response function H(ω) gives the steady state response of a system to a sine wave input. By measuring H(ω) for all frequencies, we completely define the dynamic characteristics of the system.
- An alternative approach is to measure **transient response**, initiated by a suitable disturbance. If we measure the transient response for all times, until the static equilibrium has been regained after the disturbance, then this is another method of defining completely a system's dynamic characteristics.
- It is usual to consider the result of exciting the system by an input disturbance which is short and sharp and is present for a very short (theoretically zero) time interval. The transient response is then not complicated by removal of the disturbance. Using the delta function notation, we can represent such a disturbance by the equation: $x(t) = I\,\delta(t)$

  where I is a constant parameter with the dimensions (x)x(time).

Impulse response method

- For the case when x(t) represents a force

$$x(t) = I\,\delta(t)$$

describes a hammer blow or impulse of magnitude:

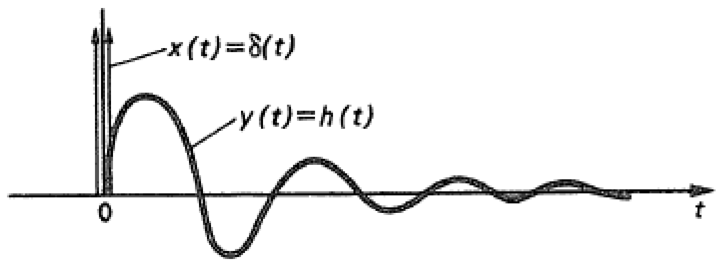
$$\int_{-\infty}^{\infty} x(t)\,\mathrm{d}t = I \int_{-\infty}^{\infty} \delta(t)\,\mathrm{d}t = I \text{ in (force)} \times \text{(time) units.}$$

- This terminology is carried over to the general case when x(t) may represent any input parameter, whether a force or not, and the impulse response of a system is defined as the system's response to an impulsive output of the form $x(t) = I\,\delta(t)$

where I has the proper dimensions. The excitation is described as a unit impulse when I is numerically unity in the above equation.

Impulse response method

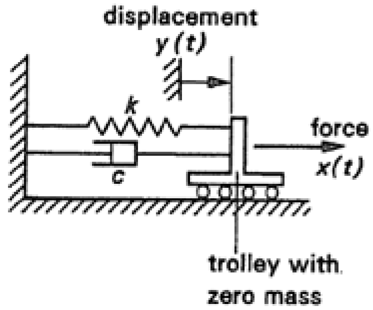
- In response to such an impulsive input, the initially dormant system suddenly springs to life and then gradually recovers its static equilibrium position as time passes.
- The response to a unit impulse at t=0 is represented by the (unit) impulse response function, h(t) as shown in the figure.



- Notice that h(t)=0 for t<0 because y(t)=0 before the impulse occurs.

Example

- Determine the impulse response function for the system shown in the figure.



- Starting from the equation of motion $c\dot{y} + ky = x,$

  h(t) is the solution y(t) when x(t)=δ(t), i.e., $c\dot{h} + kh = \delta(t).$
- For t>0, δ(t)=0 and so $c\dot{h} + kh = 0$

  whose solution is: $h = Ce^{-kt/c}$

  where C is a constant to be determined from the conditions at t=0. In order to find what happens when the hammer blow falls, we can make use of the fact that δ(t) is zero everywhere except at t=0.

Example

- If t=0- is a point just to the left and t=0+ a point just to the right of the origin, then

$$\int_{0-}^{0+} \delta(t)\,\mathrm{d}t = 1$$

and integrating both sides of the equation of motion from t=0- to t=0+ gives:

$$c\int_{0-}^{0+} \dot{h}\,\mathrm{d}t + k\int_{0-}^{0+} h\,\mathrm{d}t = \int_{0-}^{0+} \delta(t)\,\mathrm{d}t = 1.$$

- To go further, we need to employ some physical reasoning about what happens to the system when the impulse is applied. The massless trolley responds to the impulse with a sudden movement and its velocity will be infinite instantaneously. Because $\dot{h}(t) = \dot{y}(t)$ is infinite at t=0, the integral $\int_{0-}^{0+} \dot{h}\,\mathrm{d}t$ across $t = 0$ will be finite, even though the integration time from 0- to 0+ approaches zero in the limit. However, h(t) is not infinite at t=0, and so the value of the integral $\int_{0-}^{0+} h(t)\,\mathrm{d}t$ must be zero in the limit.

Example

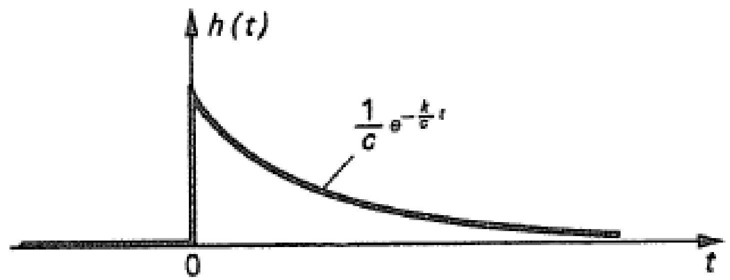
- We therefore obtain: $c\int_{0-}^{0+}\dot{h}\,\mathrm{d}t = 1$
- so that $c(h(t=0+)-0)=1$ $h(t=0+)=\frac{1}{c}.$

The full solution for h(t) is:

$$h(t) = 0 \qquad \text{for} \qquad t<0$$

$$h(t) = \frac{1}{c}\mathrm{e}^{-kt/c} \qquad \text{for} \qquad t>0$$

as sketched in the figure. Notice that the units of h(t) are those of (displacement)/(impulse) or, equivalently, of (velocity)/((force)



Unit impulse response function for the system shown in Fig.

Relationship between the frequency response and impulse response functions

- Since complete information about either the frequency response function or the impulse response function fully defines the dynamic characteristics of a system, it follows that we should be able to derive one from the other and vice versa. The Fourier transform method of breaking an aperiodic function into its frequency spectrum provides the necessary link.

- As we are dealing with stable systems which are dormant before they are excited and for which motion dies away after an impulse, we know that

$$\int_{-\infty}^{\infty} |h(t)|\,\mathrm{d}t < \infty$$

- And we may therefore take Fourier transforms of both the impulse input $x(t)=\delta(t)$ and the transient output).

Relationship between the frequency response and impulse response functions

- If we do this, we obtain:

$$X(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} x(t)\,\mathrm{e}^{-i\omega t}\,\mathrm{d}t = \frac{1}{2\pi}\int_{-\infty}^{\infty} \delta(t)\,\mathrm{e}^{-i\omega t}\,\mathrm{d}t$$

and

$$Y(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} y(t)\,\mathrm{e}^{-i\omega t}\,\mathrm{d}t = \frac{1}{2\pi}\int_{-\infty}^{\infty} h(t)\,\mathrm{e}^{-i\omega t}\,\mathrm{d}t.$$

- The first of these equations may be simplified by expanding the complex exponential to obtain:

$$X(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \delta(t)\cos\omega t\,\mathrm{d}t - i\frac{1}{2\pi}\int_{-\infty}^{\infty} \delta(t)\sin\omega t\,\mathrm{d}t$$

- and then using the property of a delta function

$$\int_{-\infty}^{+\infty}\delta(\tau - T)f(\tau)d\tau = f(\tau = T)$$

- We can show that the first integral is unity and the second integral is zero (since sinωt=0 at t=0) to obtain: $X(\omega) = \frac{1}{2\pi}$

Relationship between the frequency response and impulse response functions

- We have therefore obtained the Fourier transforms of an impulsive input x(t)=δ(t) in

$$X(\omega) = \frac{1}{2\pi}$$

and of the corresponding impulse response function y(t)=h(t) in

$$Y(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} y(t)\,\mathrm{e}^{-i\omega t}\,\mathrm{d}t = \frac{1}{2\pi}\int_{-\infty}^{\infty} h(t)\,\mathrm{e}^{-i\omega t}\,\mathrm{d}t.$$

- Furthermore, the Fourier transforms are related by the frequency response function H(ω). The nature of this relationship can be seen by the following argument. We know that when a linear system is subjected to steady state harmonic excitation at frequency ω, it responds with a steady harmonic output at the same frequency. It therefore seems reasonable to expect that, for an aperiodic input signal, frequency components X(ω)d ω in the frequency band ω to ω+d ω in the input will correspond with components Y(ω)d ω in the same frequency band in the output. In this case, if we had a harmonic input of the form $x(t) = X(\omega)\,\mathrm{d}\omega\,\mathrm{e}^{i\omega t}$

Relationship between the frequency response and impulse response functions

- The corresponding harmonic output would be given by:

$$y(t) = Y(\omega)\,\mathrm{d}\omega\, \mathrm{e}^{i\omega t}.$$

- But from $x(t) = x_0\, \mathrm{e}^{i\omega t}$

$$y(t) = H(\omega) x_0\, \mathrm{e}^{i\omega t}$$

- We also know that

$$y(t) = H(\omega) X(\omega)\,\mathrm{d}\omega\, \mathrm{e}^{i\omega t}$$

- And so, by comparing these two expressions for y(t), we obtain:

$$Y(\omega) = H(\omega) X(\omega)$$

which is a very important relation between the Fourier transforms of the input and output, X(ω) and Y(ω), and the frequency response function H(ω).

Relationship between the frequency response and impulse response functions

- Finally, substituting for X(ω) from $X(\omega)=\frac{1}{2\pi}$
- and for Y(ω) from $$Y(\omega)=\frac{1}{2\pi}\int_{-\infty}^{\infty} y(t)\,e^{-i\omega t}\,dt=\frac{1}{2\pi}\int_{-\infty}^{\infty} h(t)\,e^{-i\omega t}\,dt.$$

  into $$Y(\omega)=H(\omega)X(\omega)$$

  gives: $$\frac{1}{2\pi}\int_{-\infty}^{\infty} h(t)\,e^{-i\omega t}\,dt=H(\omega)\cdot\frac{1}{2\pi}$$

  or:

  $$H(\omega)=\int_{-\infty}^{\infty} h(t)\,e^{-i\omega t}\,dt$$

  expressing the result that the frequency response function H(ω) is the Fourier transform of the impulse response function h(t). Actually, by comparing the above equation with the definition of a Fourier transform given in

  $$X(\omega)=\frac{1}{2\pi}\int_{-\infty}^{+\infty} x(t)e^{-i\omega t}dt$$

  it can be seen that there is a ½π missing from the above equation.

Relationship between the frequency response and impulse response functions

- However, as already known, in Fourier transforms, the position of this factor is optional so long as it appears in either the Fourier transform equation or the inverse Fourier transform equation. From

$$x(t) = \int_{-\infty}^{+\infty} X(\omega) e^{i\omega t} d\omega$$

the inverse transform equation corresponding to

$$H(\omega) = \int_{-\infty}^{\infty} h(t) \mathrm{e}^{-i\omega t} \mathrm{d}t$$

is:

$$h(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} H(\omega) \mathrm{e}^{i\omega t} \mathrm{d}\omega$$

and the **impulse response equation** is shown as a Fourier integral of the **frequency response function** H(ω).

Calculation of response to an arbitrary input

- We turn now how the frequency response and impulse response functions can be used to calculate how a system responds to a prescribed excitation. Suppose that a linear system has an arbitrary input x(t) which is defined, which satisfies

$$\int_{-\infty}^{+\infty} |x(t)| dt < \infty$$

and for which we wish to calculate the resulting output y(t).

- As already described in the previous section, the frequency response function can be used to relate the Fourier Transforms of the input and output, so that if

$$X(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} x(t)\,\mathrm{e}^{-i\omega t}\,\mathrm{d}t$$

then:

$$Y(\omega) = H(\omega)X(\omega)$$

and taking the inverse transform to find y(t), we obtain:

$$y(t) = \int_{-\infty}^{\infty} H(\omega)\left\{\frac{1}{2\pi}\int_{-\infty}^{\infty} x(t)\,\mathrm{e}^{-i\omega t}\,\mathrm{d}t\right\}\mathrm{e}^{i\omega t}\,\mathrm{d}\omega$$

Calculation of response to an arbitrary input

- which is a formal solution for the output y(t). The integral with respect to ω is, in general, extremely difficult to evaluate, and so it is rarely possible to use

$$y(t) = \int_{-\infty}^{\infty} H(\omega)\left\{\frac{1}{2\pi}\int_{-\infty}^{\infty} x(t)\,\mathrm{e}^{-i\omega t}\,\mathrm{d}t\right\}\mathrm{e}^{i\omega t}\,\mathrm{d}\omega$$

  to obtain a neat closed expression for y(t). We do not therefore pursue this approach for calculating y(t) but turn to the impulse response method as a more promising alternative.
- The impulse response function h(t) gives the response at time t to a unit impulse applied at time t=0, i.e., after a delay of duration t. It follows that h(t-τ) is the response at time t to a unit impulse or “hammer blow” at time τ, i.e., after a delay of duration t-τ. Now think of an arbitrary input function x(t) as being made up of a continuous series of small impulses.

Calculation of response to an arbitrary input

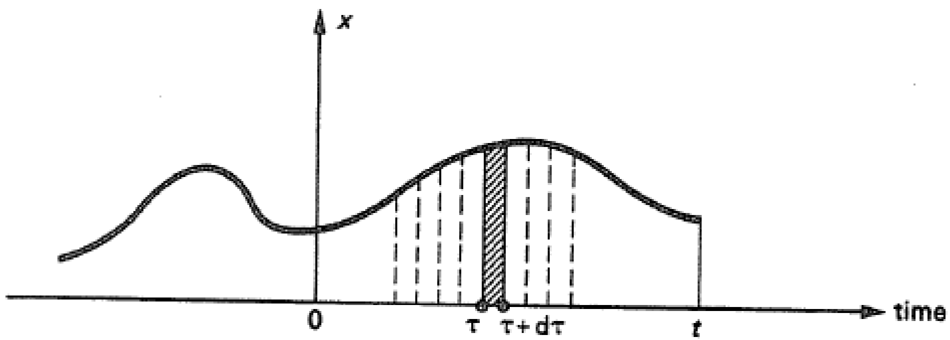
- The "impulse" corresponding to the input x(t) between the time limits τ and τ+dτ as shown in the figure has the magnitude x(τ)dτ as shown shaded. The response at time t to this "impulse" alone is just the fraction

$$\frac{x(\tau)\,d\tau}{1}$$

of the response to a unit impulse at t=τ, which is h(t-τ). The shaded area in the figure therefore contributes an amount

$$h(t-\tau)x(\tau)\,d\tau$$

to the total response at time t.



Breaking down an arbitrary input $x(t)$ into a series of "impulses"

Calculation of response to an arbitrary input

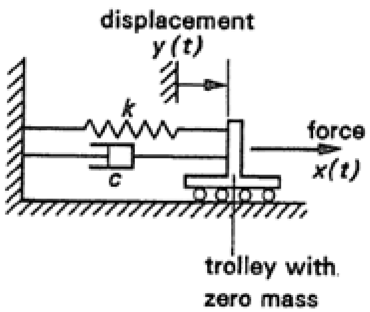
- Furthermore, since the principle of superposition applies for a linear system, we may obtain the total response y(t) at t by adding together all the separate responses to all the small “impulses” which make up the total time history back to t=-∞. We therefore integrate the response to x(τ)d τ back to τ=-∞ to give:

$$y(t) = \int_{-\infty}^{t} h(t-\tau)x(\tau)\,\mathrm{d}\tau$$

which is another formal expression for the response y(t) at time t as a result of an excitation whose value at time τ is x(τ) and which may exist from τ=-∞ to the present time τ=t.

Example

Example: Calculate the response at time t>0 of the system shown in the figure when it is subjected to a step input $x(t)=x_o$ at t=0.



From the previous example we already have

$$h(t) = 0 \qquad \text{for} \qquad t < 0$$

$$h(t) = \frac{1}{c} e^{-kt/c} \qquad \text{for} \qquad t > 0$$

so that

$$h(t-\tau) = 0 \qquad \text{for} \qquad \tau > t$$

$$h(t-\tau) = \frac{1}{c} e^{-k(t-\tau)/c} \qquad \text{for} \qquad \tau < t.$$

The excitation function is

$$x(\tau) = 0 \qquad \text{for} \qquad \tau < 0$$

$$x(\tau) = x_0 \qquad \text{for} \qquad \tau > 0$$

Example

- Using

$$y(t) = \int_{-\infty}^{t} h(t-\tau)x(\tau)\,\mathrm{d}\tau$$

$$y(t) = \int_{-\infty}^{t} h(t-\tau)x(\tau)\,\mathrm{d}\tau = \int_{0}^{t} \frac{1}{c}\mathrm{e}^{-k(t-\tau)/c}x_0\,\mathrm{d}\tau \qquad \text{for} \qquad t > 0$$

the lower limit of the integral being changed to 0 since $x(\tau) = 0$ for $\tau < 0$. Evaluating the integral gives

$$y(t) = \frac{x_0}{k}(1 - \mathrm{e}^{-kt/c}) \qquad \text{for} \qquad t > 0.$$

The response $y(t)$ is obviously zero for $t < 0$ before the step input has occurred.

- The superposition or convolution integral

$$y(t) = \int_{-\infty}^{t} h(t-\tau)x(\tau)\,\mathrm{d}\tau$$

is a most important input-output relationship for a linear system. Provided that the system is passive so that it only responds to past inputs and h(t) decays eventually to static equilibrium, so that

$$\int_{-\infty}^{\infty} |h(t)|\,\mathrm{d}t < \infty,$$

Calculation of response to an arbitrary input

- Then $y(t) = \int_{-\infty}^{t} h(t-\tau)x(\tau)\,d\tau$ applies for any input x(t) whose magnitude |x(t)| is in general bounded by a finite level. Notice that x(t) does not need to satisfy

$$\int_{-\infty}^{+\infty} |x(t)|\,dt < \infty$$

which is necessary in the classical Fourier transform (frequency response approach).

- There are three alternative versions of

$$y(t) = \int_{-\infty}^{t} h(t-\tau)x(\tau)\,d\tau$$

which can easily be derived by physical reasoning. First, we must recall that h(t-τ) is the response to a unit impulse at (t-τ)=0 that is at t=τ. For (t-τ)<0, there is no response, as there has been no impulse applied. Hence, for τ>t

$$h(t-\tau) = 0.$$

Calculation of response to an arbitrary input

- We may therefore extend the upper limit of the integral in

$$y(t) = \int_{-\infty}^{t} h(t-\tau)x(\tau)\,d\tau$$

from $\tau=t$ to $\tau=\infty$ without changing the result, since $h(t-\tau)$ is zero in this interval. The first alternative form of the above equation is therefore:

$$y(t) = \int_{-\infty}^{\infty} h(t-\tau)x(\tau)\,d\tau.$$

- Next, consider changing the variable in the first equation by putting

$$\theta = t - \tau$$

where θ may be interpreted as the time delay between the occurrence of an impulse and the instant when its result is being calculated. The limits of integration $\tau=-\infty$ and $\tau=t$ now become $\theta=\infty$ (notice that time t is a constant) and $\theta=0$ and $d\tau$ becomes $-d\theta$, so that substituting into

$$y(t) = \int_{-\infty}^{t} h(t-\tau)x(\tau)\,d\tau$$

Calculation of response to an arbitrary input

gives:

$$y(t) = \int_{\infty}^{0} h(\theta)x(t-\theta)(-\mathrm{d}\theta)$$

or changing over the limits of integration to dispense with the minus sign in front of dθ

$$y(t) = \int_{0}^{\infty} h(\theta)x(t-\theta)\,\mathrm{d}\theta.$$

- Finally, the third alternative form can be obtained either by putting θ=(t-τ) in

$$y(t) = \int_{-\infty}^{t} h(t-\tau)x(\tau)\,\mathrm{d}\tau$$

or by noting that h(θ)=0 for θ<0, since there is no response before the impulse occurs, and using the above equation, to obtain:

$$y(t) = \int_{-\infty}^{\infty} h(\theta)x(t-\theta)\,\mathrm{d}\theta.$$

Calculation of response to an arbitrary input

- Collecting together these important alternative results, the response y(t) of a passive linear system to an arbitrary input x(t) can be calculated by evaluating one of the superposition integrals:

$$y(t) = \int_{-\infty}^{t} h(t-\tau)x(\tau)\,d\tau$$

or

$$y(t) = \int_{-\infty}^{\infty} h(t-\tau)x(\tau)\,d\tau$$

or

$$y(t) = \int_{0}^{\infty} h(\theta)x(t-\theta)\,d\theta$$

or

$$y(t) = \int_{-\infty}^{\infty} h(\theta)x(t-\theta)\,d\theta$$

where h(t) is the response at time t to a unit impulse x(t)=δ(t) applied at t=0.

Calculation of response to an arbitrary input-Example

Use equation (6.26) to calculate the response of the system shown in Fig. 6.3(a) to a step input $x(t) = x_0$ at $t = 0$.

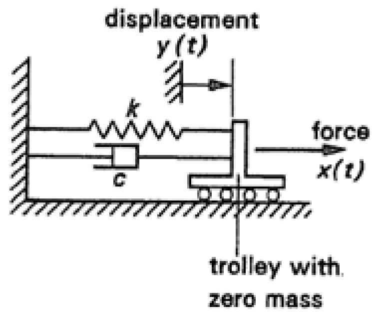
From the earlier example

$$h(\theta) = \frac{1}{c} e^{-k\theta/c} \qquad \text{for} \qquad \theta > 0$$

so that, if we want the response at time $t = t_1$ (say), then, from (6.26)

$$y(t_1) = \int_0^\infty \frac{1}{c} e^{-k\theta/c} x(t_1 - \theta)\, d\theta. \tag{6.28}$$

In order to evaluate this integral we have to determine $x(t_1 - \theta)$ as a function of θ for t_1 constant. This must be calculated from the known input function $x(t)$ shown in Fig. 6.7(a).



$$y(t) = \int_0^\infty h(\theta) x(t - \theta)\, d\theta.$$

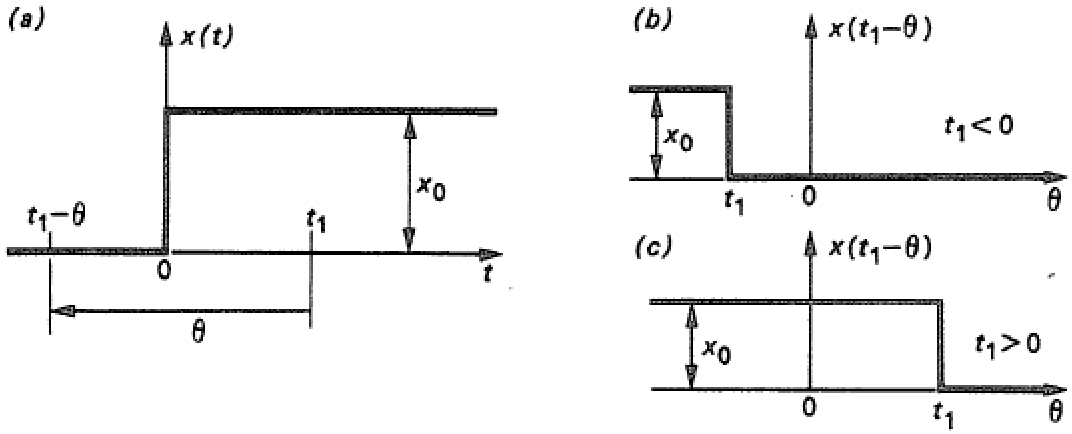


Fig. 6.7 Result of changing the variable of integration from t to $(t - \theta)$

Example

First we select the *constant* value $t = t_1$. Then, since θ is the delay between the occurrence of an "impulse" and time t_1, the variable θ is measured *backwards* along the time axis, as shown in Fig. 6.7(a). We can now read off values of $x(t_1 - \theta)$ and plot two separate Figs. 6.7(b) and (c) for the two cases $t_1 < 0$ and $t_1 > 0$.

The function $x(t_1 - \theta)$ that we need to substitute in (6.28) is therefore different for the two cases and so we obtain two different analytical results: one for the case $t_1 < 0$ and the other for the case $t_1 > 0$. These results are

$$y(t_1) = 0 \qquad \text{for} \qquad t_1 < 0$$

$$y(t_1) = \int_0^{t_1} \frac{1}{c} \mathrm{e}^{-k\theta/c} x_0 \, \mathrm{d}\theta = \frac{x_0}{k}(1 - \mathrm{e}^{-kt_1/c}) \qquad \text{for} \qquad t_1 > 0$$

in agreement with the previous example in which we applied (6.24) rather than (6.26).