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Probabilistic and stochastic methods in structural dynamics

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Course Schedule

- Introduction to probability distributions and averages
- Joint probability distributions, ensemble averages
- Correlation
- Spectral density
- Excitation response relations for linear systems
- Transmission of random vibration
- Statistics of narrow band processes
- Acuuracy of measurements

References

General vibration theory

- Rao S.S., 'Mechanical vibrations', Pearson, Prentice Hall, 2004.
- Inman D.J., 'Engineering Vibration', Prentice Hall, 1994.
- Meirovitch L., 'Fundamentals of vibrations', McGrawHill, 2001.

References

Random vibrations:

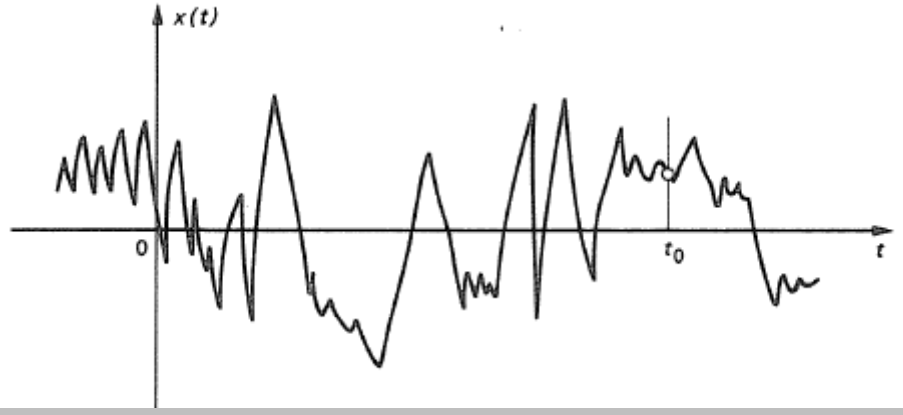
- Bendat J.S. and Piersol A.G., 'Random data analysis and measurement procedures', Wiley Series in Probability and Statistics, 3rd Edition, 2004.
- Lutes L.D. and Sarkani S., 'Random Vibrations: Analysis of structural and mechanical systems', Elsevier, 631 pp, 2004.
- Newland D.E., 'An introduction to random vibrations, spectral and wavelet analysis', Longman, 1975/1984/1993.
- Soong T.T. and Grigoriu, 'Random vibration of mechanical and structural systems', Prentice Hall, 1993.
- Wirsching P.H. and Paez T.L. and Ortiz K. 'Random vibrations: Theory and Practice', John Wiley and sons, 1995.
- Bendat J.S. and Piersol A.G. 'Engineering applications of correlation and spectral analysis', John Wiley and Sons, 2nd Edition, 1993.

Random vibrations

- A system is vibrating if it is shaking or trembling or moving backwards and forwards in some way. If this motion is unpredictable, then the system is said to be in **random vibration**.

- The subject of random vibrations is concerned with finding out how the statistical (or average) characteristics of the motion of a randomly excited system depend on the statistics of the excitation and the dynamic properties of the vibrating system.

- Figure shows part of a possible time history for a system in random vibration. The displacement x from an arbitrary datum is plotted as a function of time t. Since motion is random, the precise value of x at any chosen time $t=t_o$ cannot be precisely predicted. The bets we can do is to find the chance, or probability that x at t_o will lie within certain limits. The subject of probability is therefore at the heart of random vibration theory and we begin by considering some of the fundamental ideas of **probability theory**.



Where do we use random vibration analysis?

- Acquiring random data
- Estimating parameters of a structural system subjected to random excitation based on measured excitations and responses
- Designing structural and mechanical systems for random environments
- Modifying a structure to resist random vibrations
- Structural health monitoring

Probability density function

- Suppose first that we are dealing with a time history which is non-random or deterministic and is infact a sine wave. In this case, we can exactly predict the value of *x* for any given value of *t*. We can therefore calculate the proportion of time that the waveform spends between any two levels of *x*.

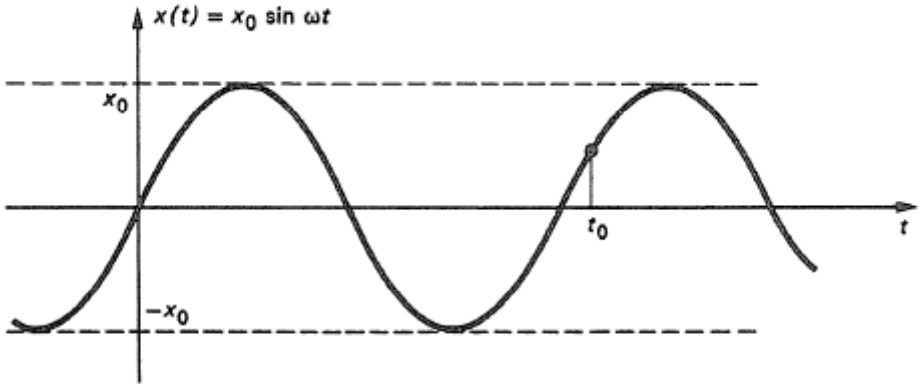


Fig. 1.2 Waveform for steady state deterministic vibration $x(t) = x_0 \sin \omega t$

Probability density function

- With reference to the figure below, during one complete cycle, x(t) lies in the band x to x+Δx for two periods of dt each. If

$$x = x_o \sin \omega t \quad \text{then} \quad dx = x_o \omega \cos \omega t dt \quad \text{so that}$$

$$dt = \frac{dx}{x_o \omega \cos \omega t}; \quad \text{Now substituting for } \cos \omega t = \sqrt{1 - \sin^2 \omega t}$$

$$dt = \frac{dx}{x_o \omega \sqrt{1 - \frac{x^2}{x_o^2}}}$$

- The proportion of time per cycle that x(t) spends in the band x to x+ Δx is therefore,

$$\frac{2(dt)}{T} = \frac{2dx}{\omega T \sqrt{\left(x_o^2 - x^2\right)}}$$

which putting $T = \frac{2\pi}{\omega}$ for the period of sine waves give:

$$\frac{2dt}{T} = \frac{dx}{\pi \sqrt{\left(x_o^2 - x^2\right)}}$$

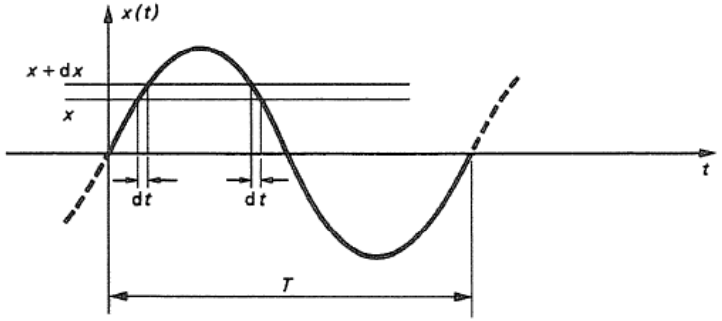


Fig. 1.3 Illustrating calculation of time for which $x \leqslant x(t) \leqslant x + dx$

Probability density function

- For any complete number of cycles, the below equation gives the proportion of the total elapsed time for which x(t) lies within the x to x+Δx band.

$$\frac{2dt}{T} = \frac{dx}{\pi\sqrt{\left(x_o^2 - x^2\right)}}$$

- Now consider a situation in which we have to choose an instant of time, t=t_o say, and find the value of x at this instant. Since x(t) is a deterministic sine wave, as soon as we specify t_o, we immediately know x(t_o). But suppose that t_o is not precisely specified. Instead we are just told that it may lie anywhere along the time axis. In this case, we can not say what x(t_o) will be; the best we can do is to predict what it may be. If t_o is chosen perfectly arbitrarily, t_o may lie anywhere during a complete cycle (the record is assumed to exist for ever with no beginning and no ending so that there is no question of t_o falling in an unfinished cycle.) The chance or probability that x(t_o) lies in the band x to x+Δx will then depend only on how long per cycle x(t) lies between x to x+Δx.

Probability density function

- The probability that $x \le x(t_o) \le x + dx$ is therefore given by:

$$\text{Prob}\left(x \le x(t_o) \le x + dx\right) = \text{Fraction of time per cycle for which x(t) lies within the x to x + dx band}$$

$$= \frac{2(dt)}{T}$$

$$= \frac{dx}{\pi\sqrt{\left(x_o^2 - x^2\right)}} \quad \text{for } -x_o \le x \le x_o$$

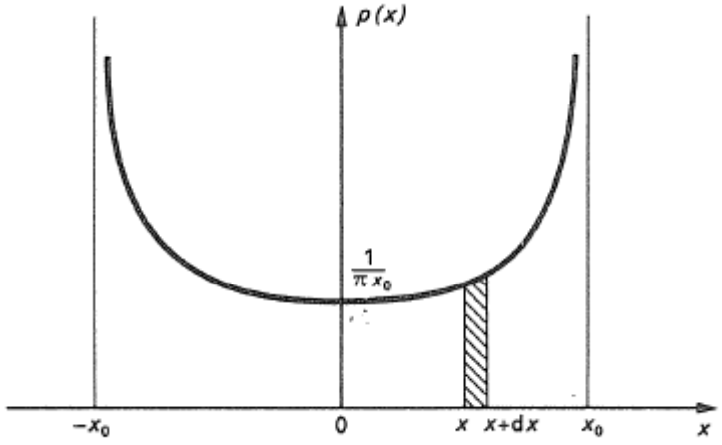
since x_o is the amplitude of motion.

- The first order probability density function p(x) is defined so that

$$\text{Prob}(x \le x(t_o) \le x + dx) = p(x)dx$$

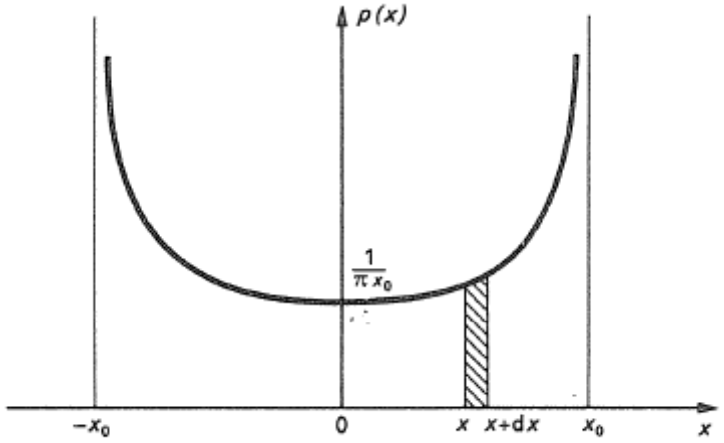
therefore in this case $p(x) = \dfrac{1}{\pi\sqrt{\left(x_o^2 - x^2\right)}}$ for $-x_o \le x \le x_o$

- This function is plotted below:



Probability density function

- Notice that this is a **probability density curve** so that it is the area under the curve which gives the probability. Hence the probability that $x(t_o)$ lies in the band x to x+dx is given by the shaded area under the curve which is:



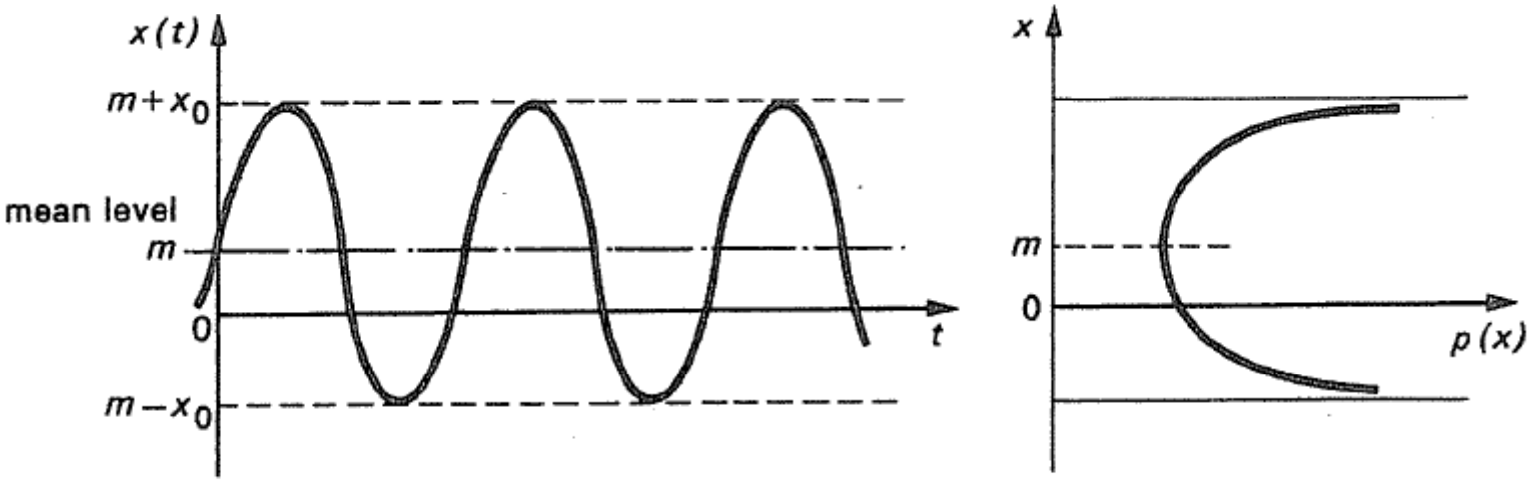
$$\text{Prob}(-x_o \le x(t_o) \le x_o) = \int_{-x_o}^{x_o} p(x)dx$$

$$= \int_{-x_o}^{x_o} \frac{1}{\pi\sqrt{\left(x_o^2 - x^2\right)}}dx$$

- Since, for the sine wave, any value x, chosen at random must lie in the range $-x_o$ to $+x_o$, the integral on the rhs of the above equation must be unity, since the probability of certainty is 1 or 100 percent.

Probability density function

- The probability density function p(x) gives the density of the distribution of values of x and it may be helpful to see this as drawn in the figure. Since a sine wave spends more of its time near its peak values than it does near its mean value, the probability density function increases towards the extremes of motion and is a minimum at the mean.



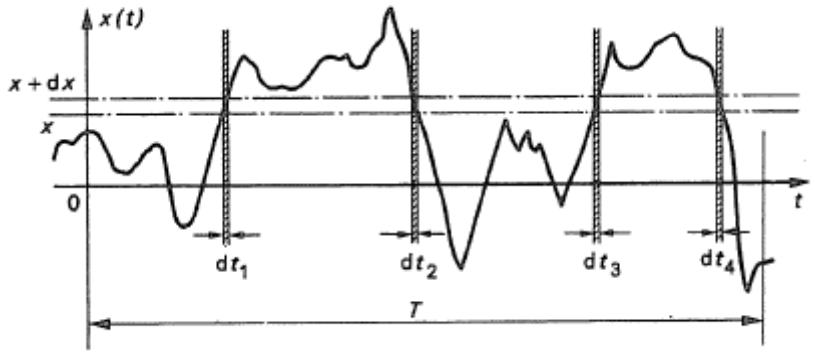
Illustrating the probability density function $p(x)$ for a sine wave superimposed on a constant mean level

Probability density function

- Now consider the situation in which x(t) is no longer a sine wave, but instead represents a random process. When we say that x(t) is random, we mean that the values of x(t) can not be precisely predicted in advance.
- Assuming that the statistical characteristics of x(t) are not changing with time, then we can use this time history to calculate the probability density function for x(t) in exactly the same way as we have just done for a deterministic function. The figure shows a sample time history for a random process with the times for which $x \le x(t) \le x + dx$ identified by the shaded strips. During the time interval T, x(t) lies in the band of values x to x+dx for a total time of $(dt_1+dt_2+dt_3+dt_4)$. We can therefore say that if T is long enough, the probability density function p(x) is given by:

$$f(x)dx = \text{Fraction of the total elapsed time for which x(t) lies in the x to x + dx band}$$

$$= \frac{(\mathrm{dt}_1 + \mathrm{dt}_2 + \mathrm{dt}_3 + \cdots)}{T} = \frac{\sum dt}{T}$$



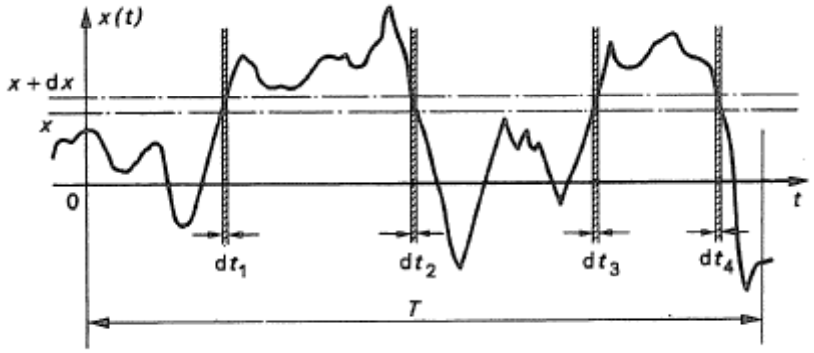
Probability density function

- For the equation

$$f(x)dx = \text{Fraction of the total elapsed time for which x(t) lies in the x to x + dx band}$$

$$= \frac{(dt_1 + dt_2 + dt_3 + \cdots)}{T} = \frac{\sum dt}{T}$$

- To be mathematically correct, the time interval T must be infinite, which means that the sample time history must go on forever and must not change its character with time.
- When x(t) is a random function of t, we can not use the above equation to calculate a mathematical expression for p(x). For any given sample time history, the only thing to do is to measure p(x) by laboriously dividing the sample record into many different levels, measuring the time spent at each band of values of x, and then using the above equation.

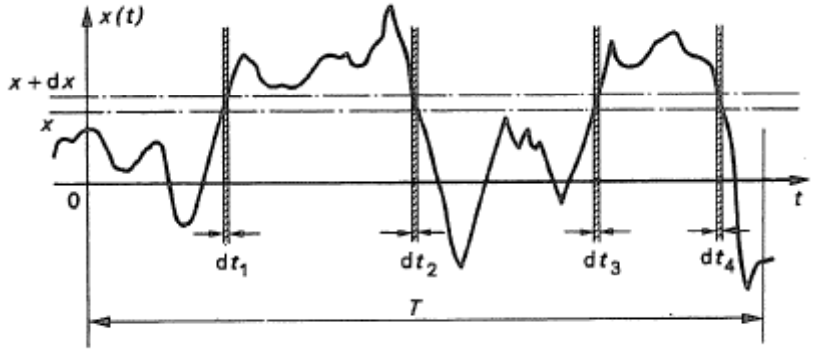


Probability density function

- Alternatively, an instrument called a **probability analyzer** will do the same thing much more quickly by sampling the time history at a series of closely spaced intervals. If there are N sampling values, and dn of these values lie in the band x to x+dx, then, the probability density function is:

$$f(x)dx = \text{Fraction of the total number of samples which lie in the x to x + dx band}$$

$$= \frac{dn}{N}$$

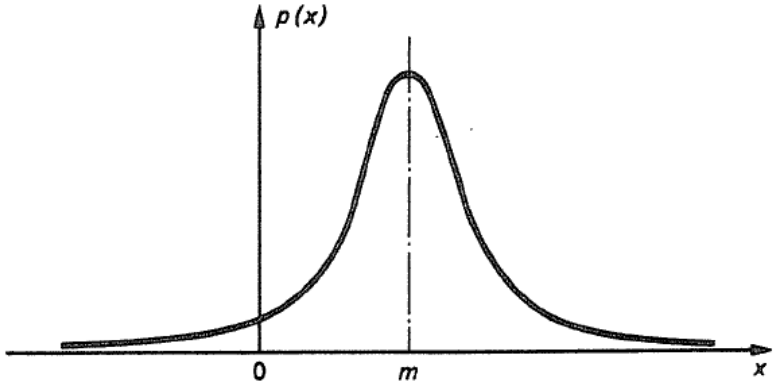


Gaussian distribution

- It is an interesting fact of life that many naturally occurring random vibrations have the well known bell shaped probability distribution shown in the figure. When the shape of the bell is given by the equation

$$p(x) = \frac{1}{\sqrt{2\pi}\,\sigma} e^{-(x-m)^2/2\sigma^2}$$

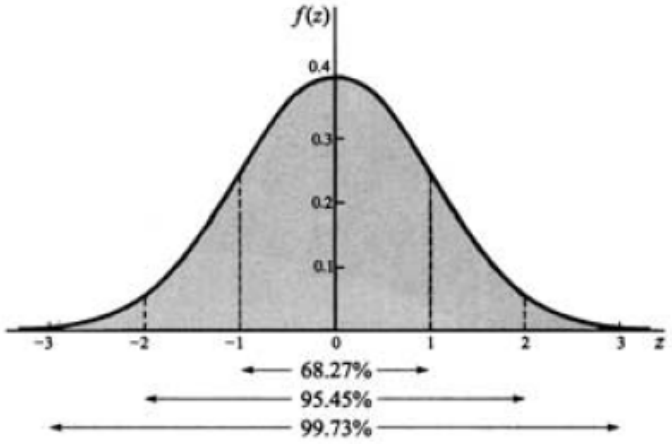
- Where m and σ are constants which we shall identify shortly, the random variable x has a normal or Gaussian probability distribution, and this distribution is extensively used in random vibration theory to approximate the characteristics of random excitation. Here σ is a measure of the spread of the function in the neighborhood of m.

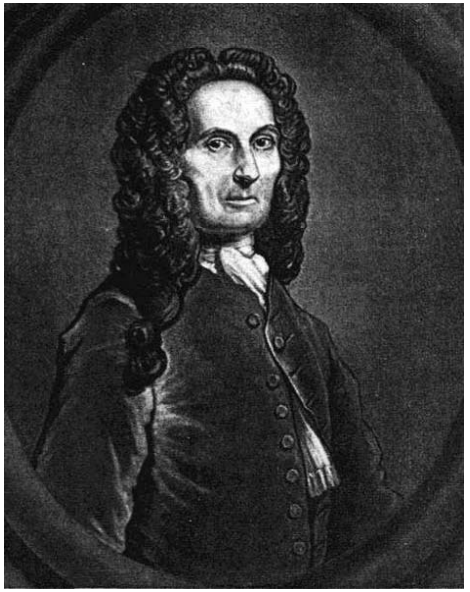


Gaussian (normal) distribution

- The Gaussian distribution is also called the **normal distribution**.

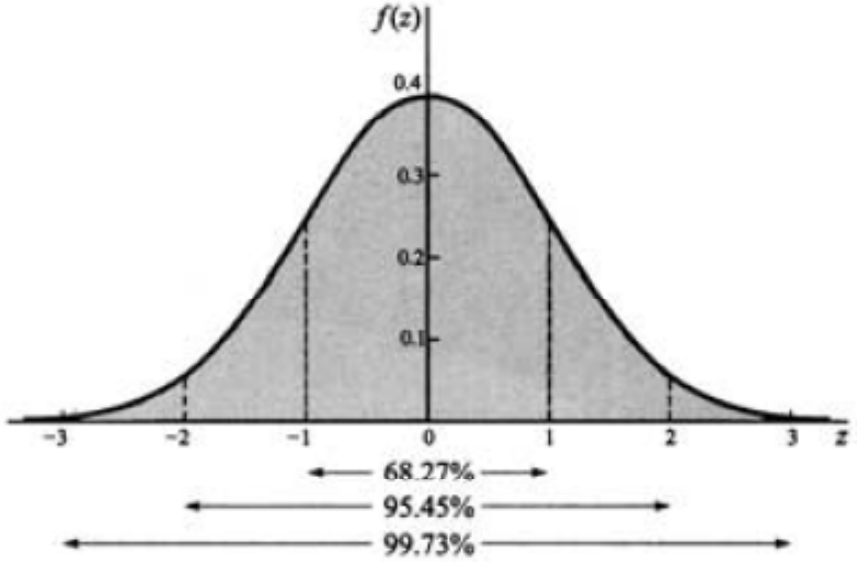
- It is believed that the first mathematician to discover some of the concepts associated with the **normal distribution** was the English mathematician, Abraham de Moivre (1667–1754). He was born in France, but moved to England because of the French government's restrictions on civil liberties.





Gaussian distribution

- He made many contributions to mathematics. In probability, he tossed a large number of coins many times and recorded the number of heads that resulted on each trial. He found that approximately 68% of the results fell within a predictable distance (now called the **standard deviation**) on either side of the mean and that 95% of the results fell within two predictable distances on either side of the mean.



Gaussian distribution

- In addition, he noticed that the shape of the distribution was bell-shaped, and he derived the equation for the normal curve in 1733, but his work in this area of mathematics went relatively unnoticed for a long period of time.
- In 1809, a German mathematician, Carl Friedrich Gauss (1777–1855) deduced that the errors in the measurements of the planets due to imperfections in the lenses in telescopes and the human eye were approximately bell-shaped. The theory was called **Gauss' Law of Error**. Gauss developed a complex measure of variation for the data and also an equation for the normal distribution curve. The curve is sometimes called the **Gaussian distribution** in his honor.

Gaussian distribution

- Around 1830, researchers began to notice that the normal distribution could be used to describe other phenomena. For example, in 1846 Adolphe Quetelet (1796–1874) began to measure the chest sizes of Scottish soldiers. He was trying to develop the concept of the ‘‘average man,’’ and found that the normal distribution curve was applicable to these measurements.

- A German experimental psychologist, Hermann Ebbinghaus (1855–1913) found that the normal distribution was applicable to measures of intelligence and memorization in humans.

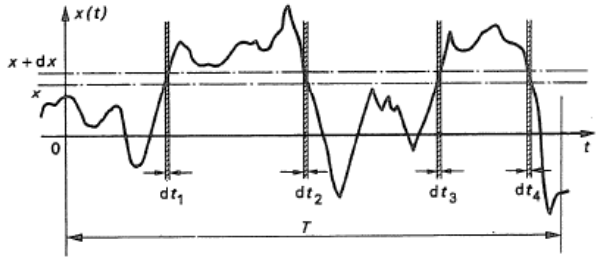
- It wasn’t until 1924 that Karl Pearson found that de Moivre had discovered the formula for the normal distribution curve long before Laplace or Gauss.

Calculation of averages

- Assuming that the probability density function p(x) is available for a random process, it can be used to calculate certain statistics of the random process x(t).
- To illustrate how this is done, we consider first the mean value of x which is usually denoted by E[x] where E stands for the statistical expectation of.
- With reference to the figure, the mean value of the time history of x over the interval T is defined so that

$$(E[x])T = \text{Total area under the } x(t) \text{ curve during the interval } T \text{ (areas below the zero line subtracting from the total area)}$$

$$= \int_0^T x(t)\,dt$$



Calculation of averages

- Hence,

$$E[x] = \int_0^T x(t)\frac{\mathrm{d}t}{T}$$

- By employing some engineering mathematics, we can now use

$$p(x)dx = \text{Fraction of the total elapsed time for which x(t) lies in the x to x + dx band}$$

$$= \frac{(\mathrm{dt}_1 + \mathrm{dt}_2 + \mathrm{dt}_3 + \cdots)}{T} = \frac{\sum dt}{T}$$

- To introduce p(x) and convert the first equation above into an important standard form from which E[x] can be calculated whenever the probability density p(x) is known. First, the right hand side of the first equation above can be expressed as a summation over time:

$$\int_0^T x(t)\frac{\mathrm{d}t}{T} = \sum_t x(t)\cdot\left(\begin{array}{l}\text{Fraction of elapsed time during which } x(t)\\ \text{lies in the band } x \text{ to } x + (\mathrm{d}x/\mathrm{d}t)\,\mathrm{d}t\end{array}\right)$$

$$= \sum_t (x)\cdot\left(\begin{array}{l}\text{Fraction of elapsed time during which } x \text{ lies}\\ \text{in the band } x \text{ to } x + \mathrm{d}x\end{array}\right)$$

Calculation of averages

- Which from

$$p(x)dx = \text{Fraction of the total elapsed time for which x(t) lies in the x to x + dx band}$$

$$= \frac{(\mathrm{dt}_1 + \mathrm{dt}_2 + \mathrm{dt}_3 + \cdots)}{T} = \frac{\sum dt}{T}$$

- Can be converted to a summation over x

$$= \sum_x (x) \cdot \left(p(x)\,\mathrm{d}x \right)$$

$$= \int_{-\infty}^{\infty} x\, p(x)\,\mathrm{d}x$$

- By a heuristic argument in which we replace an integral with respect to time by a summation and then in turn replace the summation by an integral with respect of x. Combining the above equation by

$$E[x] = \int_0^T x(t)\frac{\mathrm{d}t}{T}$$

- Gives:

$$E[x] = \int_{-\infty}^{\infty} x\, p(x)\,\mathrm{d}x$$

Calculation of averages

- The mean square value of x, $E[x^2]$ is defined as the average value of x^2 which corresponding to

$$E[x] = \int_0^T x(t)\frac{dt}{T}$$

- İs given by:

$$E[x^2] = \int_0^T x^2(t)\frac{dt}{T}$$

- And corresponding to

$$E[x] = \int_{-\infty}^{\infty} x\,p(x)\,dx$$

- By

$$E[x^2] = \int_{-\infty}^{\infty} x^2 p(x)\,dx$$

- The latter equation being the rigorous definition. Finally, the standard deviation of x, usually denoted by σ, and the variance σ^2 are defined by the equation:

$$\sigma^2 = E[(x - E[x])^2]$$

Calculation of averages

- In probability theory, the variance of a random variable is a measure of statistical dispersion.
- Whereas the mean is a way to describe the location of a distribution, the variance is a way to capture its scale or degree of being spread out.
- The unit of variance is the square of the unit of the original variable.
- The positive square root of the variance, called the standard deviation, has the same units as the original variable and can be easier to interpret for this reason.

Calculation of averages

- That is to say, the variance is the mean of the square of the deviation of x from the mean level E[x].

$$\sigma^2 = E[(x - E[x])^2]$$

- The above equation may be simplified by multiplying out the terms which are squared to give:

$$\begin{aligned}\sigma^2 &= E[x^2 - 2xE[x] + (E[x])^2] \\ &= E[x^2] - 2E[x]\cdot E[x] + (E[x])^2\end{aligned}$$

- Since the average of a sum of terms is the same as the sum of the averages of each term separately, and the average of a constant is of course just the constant. Collecting the terms:

$$\sigma^2 = E[x^2] - (E[x])^2$$

$$\text{(variance)} = \text{(standard deviation)}^2 = \{\text{Mean square} - (\text{Mean})^2\}.$$

Example

- If we apply

$$E[x] = \int_{-\infty}^{\infty} x\, p(x)\, \mathrm{d}x$$

$$E[x^2] = \int_{-\infty}^{\infty} x^2 p(x)\, \mathrm{d}x$$

- To calculate the statistics of the Gaussian process defined by

$$p(x) = \frac{1}{\sqrt{2\pi}\,\sigma} \mathrm{e}^{-(x-m)^2/2\sigma^2}$$

- We find the mean level:

$$E[x] = \int_{-\infty}^{\infty} x\, p(x)\, \mathrm{d}x = \frac{1}{\sqrt{2\pi}\,\sigma} \int_{-\infty}^{\infty} x\, \mathrm{e}^{-(x-m)^2/2\sigma^2}\, \mathrm{d}x$$

- And the mean square

$$E[x^2] = \int_{-\infty}^{\infty} x^2\, p(x)\, \mathrm{d}x = \frac{1}{\sqrt{2\pi}\,\sigma} \int_{-\infty}^{\infty} x^2\, \mathrm{e}^{-(x-m)^2/2\sigma^2}\, \mathrm{d}x,$$

- Both of which involve difficult integrals.

Example

- These may be evaluated by changing the variable of integration to y=(x-m), so that

$$E[x] = \frac{1}{\sqrt{2\pi}\,\sigma}\int_{-\infty}^{\infty}(y+m)\,\mathrm{e}^{-y^2/2\sigma^2}\,\mathrm{d}y$$

$$E[x^2] = \frac{1}{\sqrt{2\pi}\,\sigma}\int_{-\infty}^{\infty}(y+m)^2\,\mathrm{e}^{-y^2/2\sigma^2}\,\mathrm{d}y$$

- And then using the standard results

$$\int_0^{\infty}\mathrm{e}^{-y^2/2\sigma^2}\,\mathrm{d}y = \sqrt{\frac{\pi}{2}}\sigma$$

$$\int_0^{\infty} y\,\mathrm{e}^{-y^2/2\sigma^2}\,\mathrm{d}y = \sigma^2$$

$$\int_0^{\infty} y^2\,\mathrm{e}^{-y^2/2\sigma^2}\,\mathrm{d}y = \sqrt{\frac{\pi}{2}}\sigma^3$$

- To obtain:

$$E[x] = m$$

$$E[x^2] = \sigma^2 + m^2.$$

Probability distribution function

- In addition to using the probability density function p(x) to describe the distribution of values of a random variable, the closely related probability distribution function P(x) is also often referred to and is usually computed by a probability analyzer. P(x) is defined by the equation:

$$P(x) = \int_{-\infty}^{x} p(x)\,\mathrm{d}x$$

and may therefore be interpreted as the shaded area under the probability density curve shown.

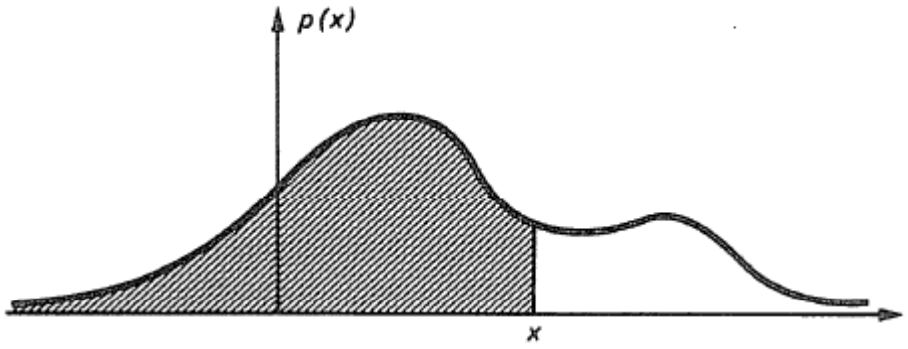


Fig. 1.9 The shaded area gives the value of the probability distribution function $P(x)$

Probability distribution function

- The value of P(x) lies between zero and unity since

$$\text{Prob}\,(-\infty \leqslant x \leqslant \infty) = \int_{-\infty}^{\infty} p(x)\,\mathrm{d}x = P(x = \infty) = 1$$

and gives the probability that a sample value is less than x. It is apparent that the equation might lead to some confusion because the variable of integration and the upper limit of the integral are the same. Therefore,

$$P(x) = \int_{-\infty}^{x} p(\xi)\,\mathrm{d}\xi$$

where ξ is a dummy variable chosen so that

$$p(\xi = x) = p(x).$$

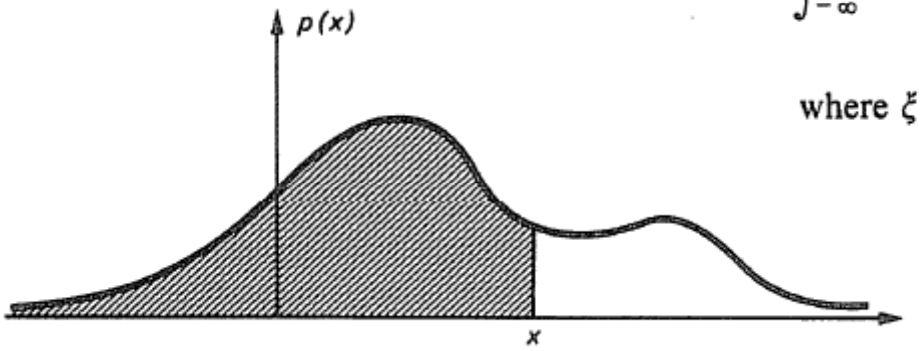


Fig. 1.9 The shaded area gives the value of the probability distribution function $P(x)$

Probability distribution function

- By differentiating with respect to x, we can obtain another final relationship:

$$\frac{\mathrm{d}P(x)}{\mathrm{d}x} = p(x)$$

which says that the slope of the probability distribution function is equal to the probability density function.

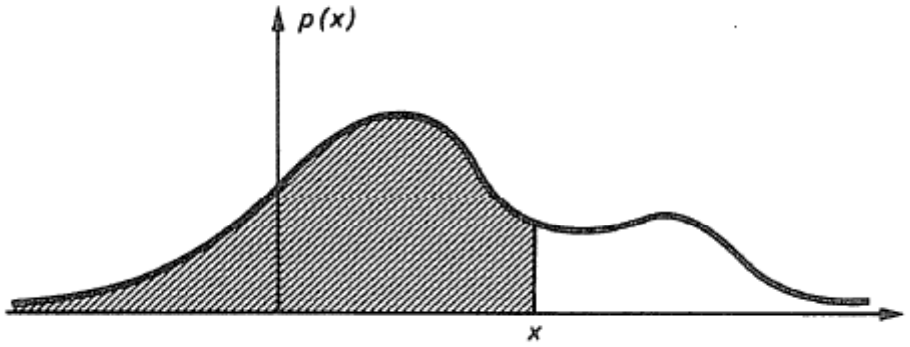
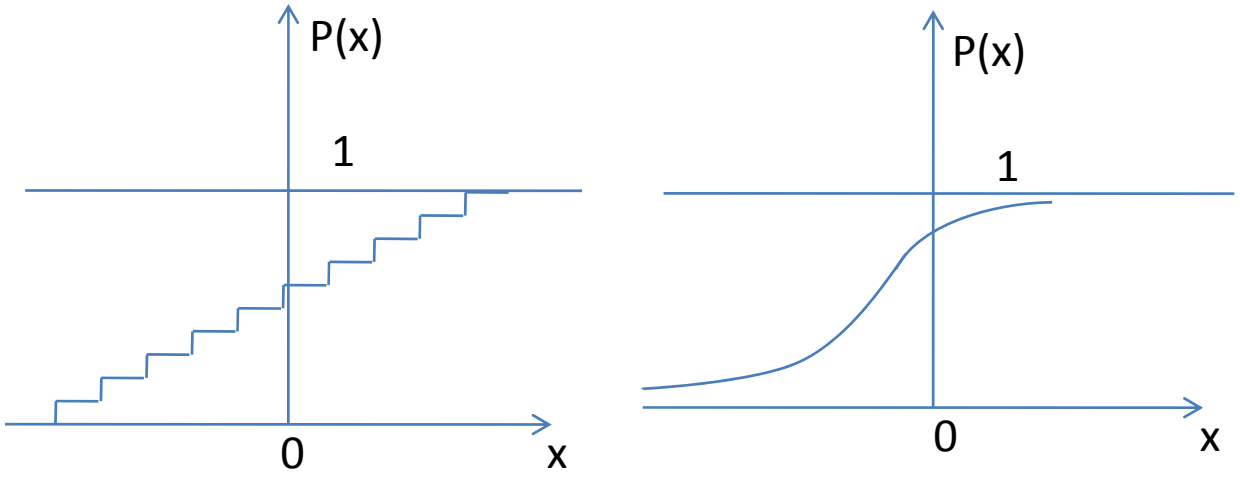


Fig. 1.9 The shaded area gives the value of the probability distribution function $P(x)$

Probability distribution function

- Probability distribution functions for a discrete random variable x and a continuous random variable x are shown in the figure below:



a) Probability distribution function for a discrete random variable x

b) Probability distribution function for a continuous random variable x

Dirac delta function

- The Dirac delta function is a function that is the limiting case of a ractangle of width Δx and height $1/\Delta x$ as Δx tends to zero.
- Due to this definition, it is clear that the area under the function remains constant and equal to 1.
- The Dirac delta function is denoted by $\delta(x-a)$ and for practical purposes, is zero everywhere except at x=a, where it is singular.
- It is likewise possible to treat discrete and continuous random variables together in a unified manner using Dirac delta functions. If a random variable X has a sample realization which is equal to x, and if X takes on a discrete value, say a, with finite probability $p_X(a)$, then the product $p_X(a)$ times $\delta(x-a)$ is added to the probability density function. The condition that the total area under the density function is equal to 1 still applies.

Probability distribution function

- For discrete random variables, the probability distribution function has the appearance of a staircase because there are discrete jumps at each value of x that has a nonzero probability; the heights of the steps are equal to the probabilities.

- It should be noted that the derivatives of these functions are singular at the discrete jumps.

- As mentioned in the previous slides, the density function may be written using Dirac delta functions to represent these instantaneous jumps.

- For continuous random variables, the probability distribution function is smooth in appearance.

Joint probability distributions, ensemble averages

Second-order probability density functions

- The first order probability density function p(x) specifies the probability p(x)dx that a random variable lies in the range of values x to x+dx.

- The second-order probability density function p(x,y) is defined in the same way but extends the number of random variables from one to two, in this case x and y.

- The probability that the x random variable lies in the range x to x+dx and that the y random variable lies in the range y to y+dy is given by p(x,y)dxdy.

Joint probability distributions, ensemble averages

Second-order probability density functions

- For instance if x(t) is one random function of time t and y(t) is another random function of t, and if both functions are sampled at time t_o, then the joint probability that

$$\text{Prob}\,(x \leqslant x(t_0) \leqslant x + \mathrm{d}x \quad \textit{and} \quad y \leqslant y(t_0) \leqslant y + \mathrm{d}y) = p(x, y)\,\mathrm{d}x\,\mathrm{d}y$$

  from the above definition of the second order probability density function p(x,y).
- In order to determine the joint probability that $x(t_o)$ and $y(t_o)$ lie within finite bands of values of x and y so that

$$\text{Prob}\,(x_1 \leqslant x(t_0) \leqslant x_2 \;\; \textit{and} \;\; y_1 \leqslant y(t_0) \leqslant y_2) = \int_{x_1}^{x_2} \int_{y_1}^{y_2} p(x, y)\,\mathrm{d}x\,\mathrm{d}y.$$

Joint probability distributions, ensemble averages

- If both bands of values extend from $-\infty$ to $+\infty$, there is a 100 percent certainty that both $x(t_o)$ and $y(t_o)$ must lie within these bounds, and so

$$\text{Prob}\,(-\infty \leqslant x(t_0) \leqslant \infty \text{ and } -\infty \leqslant y(t_0) \leqslant \infty) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} p(x,y)\,dx\,dy = 1.$$

- The joint probability density function p(x,y) may therefore be represented as a two dimensional surface for which the volume contained underneath the surface is unity (dimensionless). It follows that the dimensions of p(x,y) must be the dimensions of 1/xy.

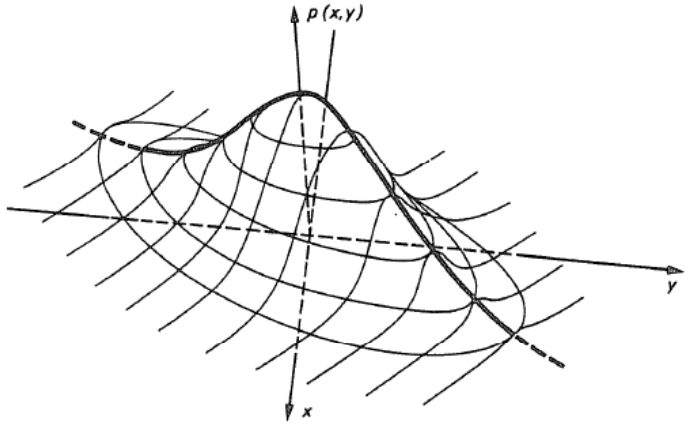


Fig. 2.1 Typical second-order or joint probability density function $p(x,y)$

Joint probability distributions, ensemble averages

- This definition of joint probability density function implies that the total volume between the xy plane and the p(x,y) surface equal unity, that is,

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} p(x,y)\,dx\,dy = 1.$$

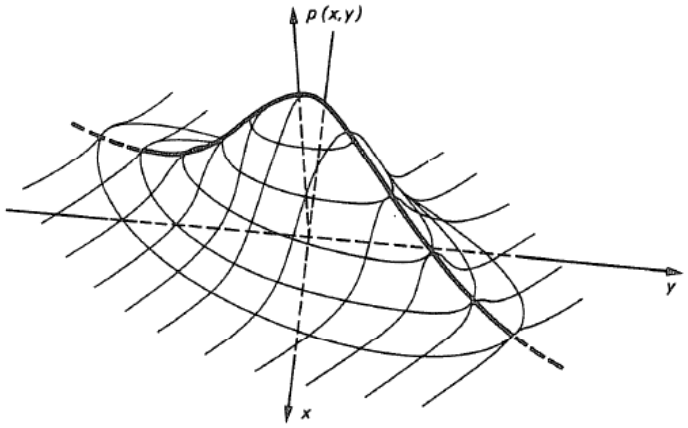


Fig. 2.1 Typical second-order or joint probability density function $p(x,y)$

Joint probability distributions, ensemble averages

- Consider now the probability of finding the random variable $x(t_o)$ within the x to x+dx band independently of the value of the random variable $y(t_o)$. This probability can be expressed in terms of the first order probability density function p(x) as:

$$\text{Prob}\,(x \leqslant x(t_0) \leqslant x + \mathrm{d}x) = p(x)\,\mathrm{d}x.$$

- But the same result must be obtained by using the second-order probability density function p(x,y) and allowing the band of values for $y(t_o)$ to extend from $-\infty$ to $+\infty$. From

$$\text{Prob}\,(x_1 \leqslant x(t_0) \leqslant x_2 \;\; and \;\; y_1 \leqslant y(t_0) \leqslant y_2) = \int_{x_1}^{x_2}\int_{y_1}^{y_2} p(x,y)\,\mathrm{d}x\,\mathrm{d}y.$$

we obtain:

$$\text{Prob}\,(x \leqslant x(t_0) \leqslant x + \mathrm{d}x \;\; and \;\; -\infty \leqslant y(t_0) \leqslant \infty) = \mathrm{d}x \int_{-\infty}^{\infty} p(x,y)\,\mathrm{d}y.$$

Since both of the above equations are both expressing the same result, it follows that the rhs of each equation is the same, so that

Joint probability distributions, ensemble averages

$$p(x)\,\mathrm{d}x = \mathrm{d}x \int_{-\infty}^{\infty} p(x,y)\,\mathrm{d}y$$

or

$$p(x) = \int_{-\infty}^{\infty} p(x,y)\,\mathrm{d}y.$$

The first order probability density functions can therefore be obtained from the corresponding joint probability density function by integrating out the dependence of the unwanted random variable according to the above equation. By comparison with the above equation, the first-order probability density function for y is thus given by:

$$p(y) = \int_{-\infty}^{\infty} p(x,y)\,\mathrm{d}x.$$

Second-order averages

- In order to determine the average value of a quantity which is a function of two random variables, we return first to the equations E[x] and E[x^2]. These are:

$$E[x] = \int_{-\infty}^{\infty} x\, p(x)\, \mathrm{d}x$$

$$E[x^2] = \int_{-\infty}^{\infty} x^2\, p(x)\, \mathrm{d}x.$$

They may be interpreted as

$$E\begin{bmatrix}\text{function of}\\ \text{a random}\\ \text{variable } x\end{bmatrix} = \sum_{\text{all } x} \begin{pmatrix}\text{value of function}\\ \text{when } x \text{ lies in the}\\ \text{band } x \to x + \mathrm{d}x\end{pmatrix} \begin{pmatrix}\text{probability that}\\ x \text{ lies in the band}\\ x \to x + \mathrm{d}x\end{pmatrix}$$

which may be extended to include the case of two random variables x and y, and then reads:

$$E\begin{bmatrix}\text{function of}\\ \text{random}\\ \text{variables } x\\ \text{and } y\end{bmatrix} = \sum_{\substack{\text{all } x\\ \text{and}\\ \text{all } y}} \begin{pmatrix}\text{value of function}\\ \text{when } x \text{ lies in the}\\ \text{band } x \to x + \mathrm{d}x\\ \textit{and } y \text{ lies in the}\\ \text{band } y \to y + \mathrm{d}y\end{pmatrix} \begin{pmatrix}\text{probability that}\\ x \text{ lies in the}\\ \text{band } x \to x + \mathrm{d}x\\ \textit{and } y \text{ lies in the}\\ \text{band } y \to y + \mathrm{d}y\end{pmatrix}$$

Second-order averages

- In mathematical terms, the latter becomes:

$$E[f(x,y)] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\,p(x,y)\,\mathrm{d}x\,\mathrm{d}y$$

and this is a general result for calculating the average value E[f(x,y)] of a function f(x,y) of two random variables x and y whose joint probability density function is p(x,y).

Example: In order to illustrate these ideas we shall again turn to the analysis of sine wave time histories. In this case, suppose that there are two sine waves of the same (constant) amplitude and frequency which are out of phase with each other by an unknown random phase angle ϕ so that if

$$x(t) = x_0 \sin \omega t$$

$$y(t) = x_0 \sin (\omega t + \phi).$$

Second-order averages

- It is required to calculate the mean value of the product xy and $(xy)^2$. Suppose that both time histories are sampled at the same arbitrary instant of time t_o. In this case,

$$x(t_0) = x_0 \sin(\omega t_0)$$

$$y(t_0) = x_0 \sin(\omega t_0 + \phi).$$

- Since the time t_o is arbitrary it may be anywhere along the time axis, but since the sine waves are periodic with period $T=2\pi/\omega$ we need only consider the cases for which

$$0 \leqslant t_0 \leqslant \frac{2\pi}{\omega}.$$

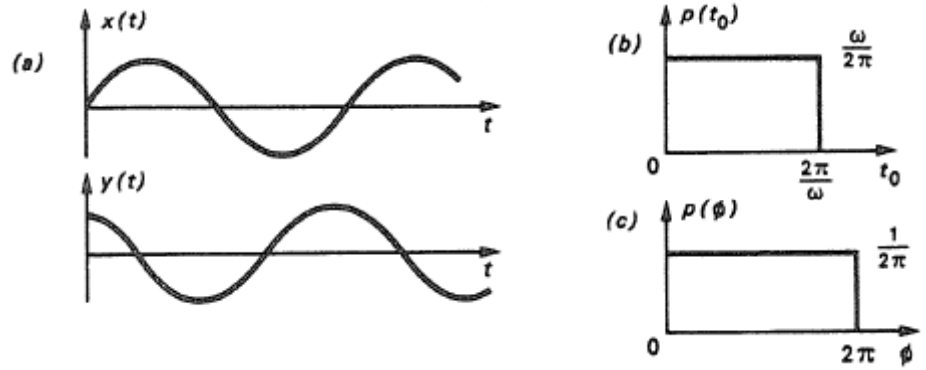
- If t_o is chosen truly arbitrarily, i.e. it is a random variable which may fall with uniform probability anywhere on the time axis, then the first order probability density function $p(t_o)$ for t_o will be as shown in the figure next slide.

Second-order averages

- Notice that, in both cases, the height of the probability density curve must be scaled so that according to

$$\text{Prob}\,(-\infty \leqslant x \leqslant \infty) = \int_{-\infty}^{\infty} p(x)\,\mathrm{d}x = P(x = \infty) = 1$$

the area under the curve is unity.



From the basic definitions we can now write

$$\text{Prob}\left(\begin{matrix}\text{time of sampling lies}\\ \text{in the band } t_0 \text{ to } t_0 + \mathrm{d}t_0\end{matrix}\right) = p(t_0)\,\mathrm{d}t_0 = \frac{\omega}{2\pi}\,\mathrm{d}t_0$$

and

$$\text{Prob}\left(\begin{matrix}\text{phase difference lies}\\ \text{in the band } \phi \text{ to } \phi + \mathrm{d}\phi\end{matrix}\right) = p(\phi)\,\mathrm{d}\phi = \frac{1}{2\pi}\,\mathrm{d}\phi.$$

Second-order averages

- To find the joint probability of two events t_o and ϕ we have to multiply together the probability of each event separately.

- This conclusion may easily be verified by thinking what happens when throwing a dice. The probability of a 4 (say) at the first throw is 1/6. The probability of a 4 at the second throw is also 1/6. The probability of two 4's in succession is however 1/36 since there are 36 possible combination of numbers from throwing the dice twice and only one of these combinations gives two 4's.

- Hence the joint probability

$$\text{Prob}\left(\begin{array}{l}\text{sampling time in the}\\ \text{band } t_0 \to t_0 + \mathrm{d}t_0\\ \textit{and} \text{ phase difference}\\ \text{in the band } \phi \to \phi + \mathrm{d}\phi\end{array}\right) = \frac{\omega}{(2\pi)^2}\,\mathrm{d}t_0\,\mathrm{d}\phi$$

Second-order averages

- The joint probability density function is

$$p(t_0, \phi) = \begin{cases} \dfrac{\omega}{(2\pi)^2} & \text{for } \begin{cases} 0 \leqslant t_0 \leqslant 2\pi/\omega \\ 0 \leqslant \phi \leqslant 2\pi \end{cases} \\ 0 \text{ for all values of } t_0 \text{ and } \phi \text{ outside these limits.} \end{cases}$$

- From

$$E[f(x,y)] = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x,y)\, p(x,y)\, \mathrm{d}x\, \mathrm{d}y$$

the average value of the product xy is:

$$E[xy] = \int_0^{2\pi/\omega} \mathrm{d}t_0 \int_0^{2\pi} \mathrm{d}\phi\, x_0^2 \sin \omega t_0 \sin(\omega t_0 + \phi) \frac{\omega}{(2\pi)^2}$$

$$= x_0^2 \frac{\omega}{(2\pi)^2} \int_0^{2\pi/\omega} \mathrm{d}t_0 \sin \omega t_0 \int_0^{2\pi} \mathrm{d}\phi \sin(\omega t_0 + \phi).$$

- Taking the right hand integral first, the integration is over ϕ with t_o held constant, and the result is zero. Hence:

$$E[xy] = 0.$$

Second-order averages

- The average value of $(xy)^2$ is:

$$E[x^2y^2] = \int_0^{2\pi/\omega} \mathrm{d}t_0 \int_0^{2\pi} \mathrm{d}\phi \; x_0^4 \sin^2 \omega t_0 \sin^2(\omega t_0 + \phi) \frac{\omega}{(2\pi)^2}$$
$$= x_0^4 \frac{\omega}{(2\pi)^2} \int_0^{2\pi/\omega} \mathrm{d}t_0 \sin^2 \omega t_0 \int_0^{2\pi} \mathrm{d}\phi \sin^2(\omega t_0 + \phi).$$

- The right hand integral is again with respect to ϕ with t_o held constant, and the result is π independently of t_o . The left hand integral may be evaluated and has the value of π/ω. Hence:

$$E[x^2y^2] = x_0^4 \cdot \frac{\omega}{(2\pi)^2} \cdot \frac{\pi}{\omega} \cdot \pi$$
$$= \tfrac{1}{4} x_0^4.$$

Conditional probability

- We have already shown how the first order probability density functions can be obtained from the corresponding second-order probability density function using the equations:

$$p(x) = \int_{-\infty}^{\infty} p(x, y)\,\mathrm{d}y. \qquad p(y) = \int_{-\infty}^{\infty} p(x, y)\,\mathrm{d}x.$$

- The first equation above for instance specifies the probability density for x to lie in the band x to x+dx independently of the value of y. Suppose now that instead of allowing the random variable y to have any value, we restrict our attention only to those cases for which y lies in the band of values y to y+dy. On the condition that y lies in the band, we want to find the distribution of x.

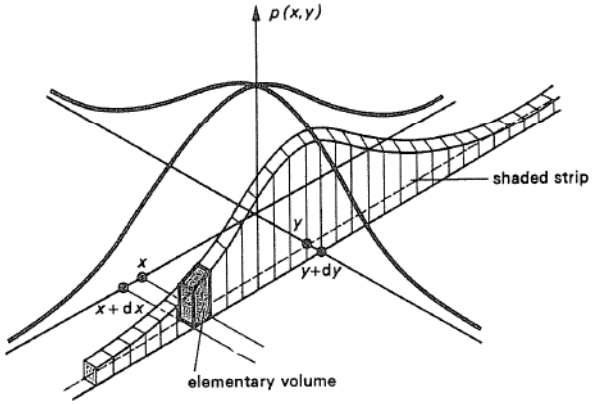
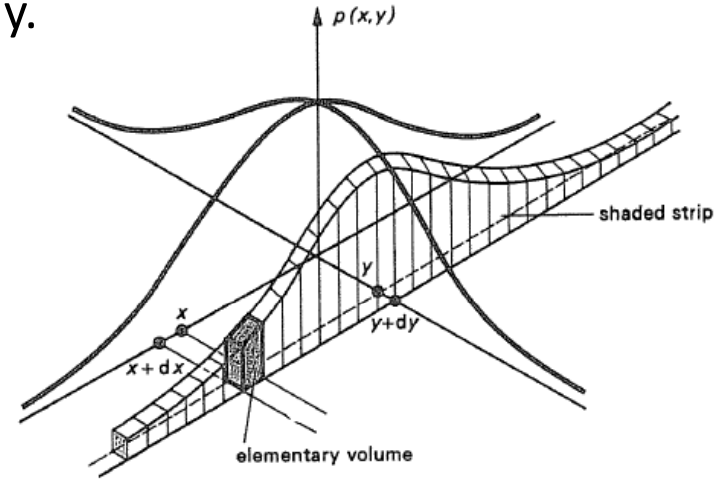


Fig. 2.3 Calculation of the conditional probability density function $p(x|y)$

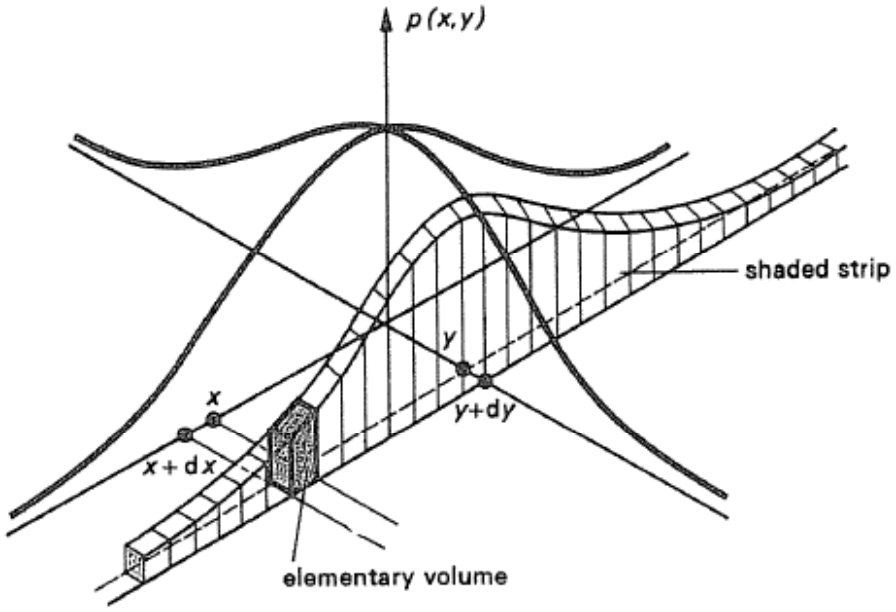
Conditional probability

- The meaning of this restriction may be seen by thinking again of the two random functions x(t) and y(t) which are sampled at arbitrary time t_o. By sampling the time histories many times (by changing t_o arbitrarily) we obtain a population of sample values of x and y. From this population, we select only those samples for which y lies in the range y to y+dy, and by so doing obtain a much reduced population.
- We write a conditional probability density function as follows, using a vertical bar,| read as “given” to set off the conditioning function. This is usually denoted as p(x|y) which means the conditional probability density function for x given y.



Conditional probability

- We can in fact determine p(x|y) from the second order probability distribution. Figure shows the joint probability density p(x,y) as a function of x and y. The elementary volume shown represents the probability that x lies in the x to x+dx band and that y lies in the y to y+dy band. The volume of the shaded strip shown represents the probability that $-\infty \leqslant x \leqslant \infty$ *and* that y lies in the y to y+dy band. Hence the conditional probability that given y lies in the y to y+dy band, x lies in the x to x+dx band, is:



$$p(x|y)\,dx = \frac{\text{elementary volume}}{\text{volume of shaded strip}}.$$

Conditional probability

- Substituting for the volumes on the r.h.s. of the equation

$$p(x|y)\,dx = \frac{\text{elementary volume}}{\text{volume of shaded strip}}.$$

gives:

$$p(x|y)\,dx = \frac{p(x,y)\,dx\,dy}{dy\int_{-\infty}^{\infty} p(x,y)\,dx}$$

which after simplifying and using $p(y) = \int_{-\infty}^{\infty} p(x,y)\,dx.$ gives:

$$p(x|y) = \frac{p(x,y)}{p(y)}.$$

- This result is important because, if the conditional probability distribution for x is independent of y then we must have:

$$p(x|y) = p(x)$$

- From which we obtain:

$$p(x,y) = p(x)p(y).$$

Conditional probability

- Random variables which satisfy $p(x,y) = p(x)p(y).$ are statistically independent variables. The physical explanation of this is that when the random variables x and y are sampled, the sampled values of x are not affected by the corresponding sampled values of y and vice versa.

- It is apparent that the conditional probability density functions are ratios of joint probability density function and the first order probability density function.

- This is the condition for the statistical independence of x and y. If x and y are two statistically independent random variables, then the joint probability density function p(x,y) can be factorized into two independent functions of x only and y only which are the first order probability density functions for x and y.

Second order Gaussian distribution

Second order Gaussian distribution

- The full definition of a multi dimensional Gaussian distribution includes, not only the first order probability density function, but also corresponding second and higher order joint densities.
- The general expression for the second-order probability density function for two jointly Gaussian random variables x and y is:

$$p(x,y) = \frac{1}{2\pi\sigma_x\sigma_y\sqrt{(1-\rho_{xy}^2)}}\, e^{-\frac{1}{2(1-\rho_{xy}^2)}\left\{\frac{(x-m_x)^2}{\sigma_x^2}+\frac{(y-m_y)^2}{\sigma_y^2}-\frac{2\rho_{xy}(x-m_x)(y-m_y)}{\sigma_x\sigma_y}\right\}}$$

where m_x and m_y are the mean values of x and y, σ_x^2 and σ_y^2 are the variances of x and y and ρ_{xy} is a correlation coefficient defined as

and called normalized covariance

$$\rho_{xy} = \frac{E[(x-m_x)(y-m_y)]}{\sigma_x\sigma_y}$$

- Notice that if ρ_{xy} =0, x and y are statistically independent because the equation

$$p(x,y) = \frac{1}{2\pi\sigma_x\sigma_y\sqrt{(1-\rho_{xy}^2)}}\, e^{-\frac{1}{2(1-\rho_{xy}^2)}\left\{\frac{(x-m_x)^2}{\sigma_x^2}+\frac{(y-m_y)^2}{\sigma_y^2}-\frac{2\rho_{xy}(x-m_x)(y-m_y)}{\sigma_x\sigma_y}\right\}}$$

may be factorized to give $$p(x,y) = \left(\frac{1}{\sqrt{2\pi}\,\sigma_x} e^{-(x-m_x)^2/2\sigma_x^2}\right)\cdot\left(\frac{1}{\sqrt{2\pi}\,\sigma_y} e^{-(y-m_y)^2/2\sigma_y^2}\right) = p(x)\cdot p(y).$$

Ensemble averaging

- So far, we have been concerned with calculating averages for a single time history or in this chapter for two time histories of different events. Our mental picture is that of sampling the continuous time history at many points and so obtaining a representative population of sample values.
- However, mathematically, there is an alternative way of calculating averages. This is based on the concept of an ensemble or collection of sample functions $x_1(t)$, $x_2(t)$,$x_3(t)$, etc., which together make up the random process x(t).

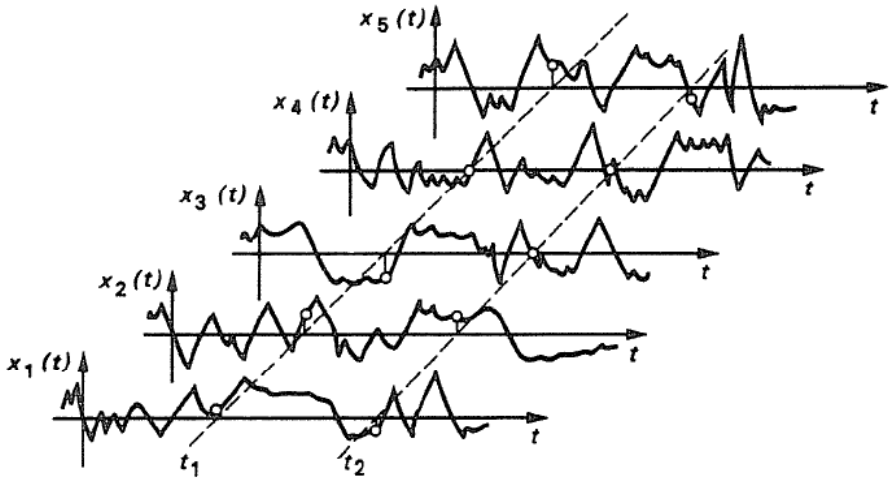


Fig. 2.4 Concept of ensemble averaging

Random processes

- A random process is an ensemble of n random variables related to a similar phenomenon which may be functions of one or more independent variables (Clough and Penzien, 1993).
- Response of a structure to a random excitation are also random processes.

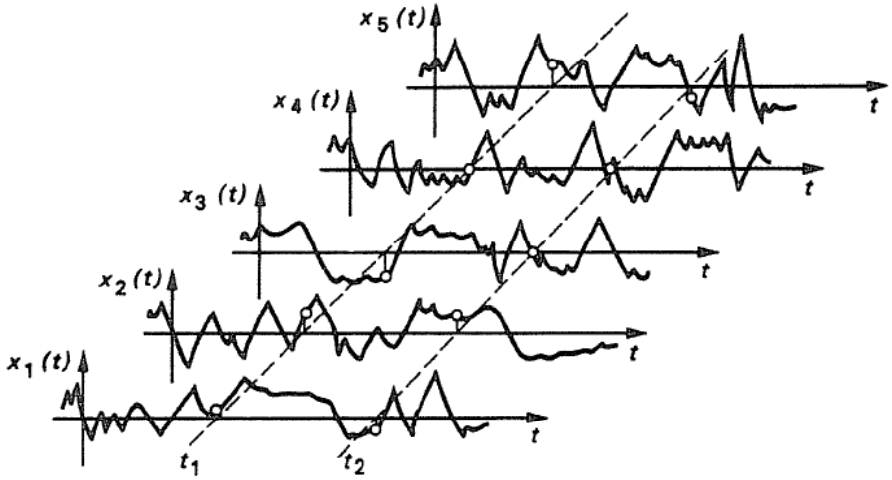


Fig. 2.4 Concept of ensemble averaging

Ensemble averaging

- Instead of being measured along a single sample, ensemble averages are measured across the ensemble.
- By determining the values of enough sample functions at time t_1, the first order probability distribution for x at t_1 can be calculated.
- If another series of measurements is made at time t_2, the second order probability distribution for x at t_1 and x at t_2 can be found.
- Similarly, by making measurements at other times, higher order probability distributions for the ensemble can be found.

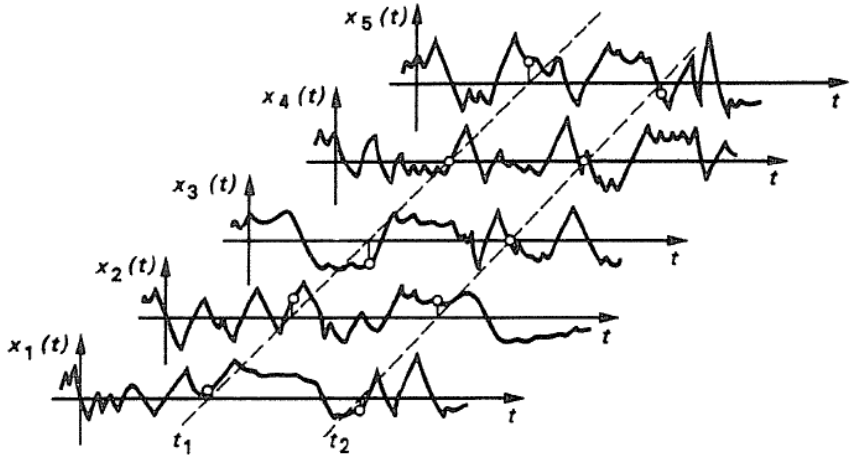


Fig. 2.4 Concept of ensemble averaging

Ensemble averaging

- The random process is said to be stationary if the probability distributions obtained for the ensemble do not depend on absolute time (but only, for second and higher order probability distributions, on the time separation between measuring points).
- Ofcourse, the term stationary refers to the probability distributions and not to the samples themselves.
- This implies that all the averages are independent of absolute time and specifically that the mean, meansquare, variance and standard deviation are independent of time altogether.

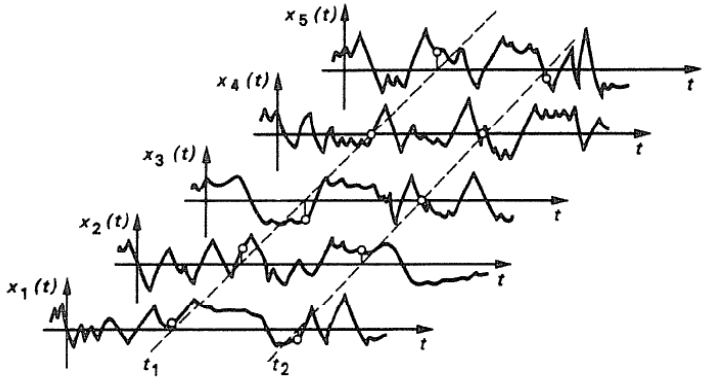


Fig. 2.4 Concept of ensemble averaging

Ensemble averaging

- Since all engineering random processes must have a beginning and ending, they can not be truly stationary, but for practical purposes it is very often adequate to assume that a process is stationary for the majority of its lifetime, or that it can be divided into several separate periods each of which is approximately stationary.
- The term weakly stationary is sometimes used to describe processes in which only the first and second order probability distributions are invariant with time, a strictly stationary process is one for which all probability distributions of the ensemble are invariant with time.

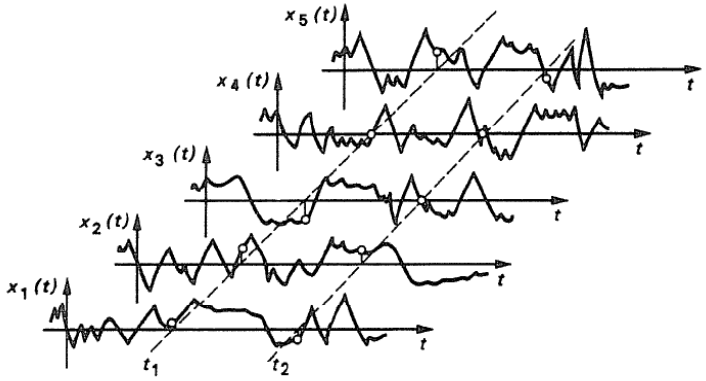


Fig. 2.4 Concept of ensemble averaging

Ensemble averaging

- A stationary process is called an ergodic process if in addition to all the ensemble averages being stationary with respect to a change of the time scale, the averages taken along any single sample are the same as the ensemble averages.
- In practical terms, each sample function is then completely representative of the ensemble that constitutes the random process.

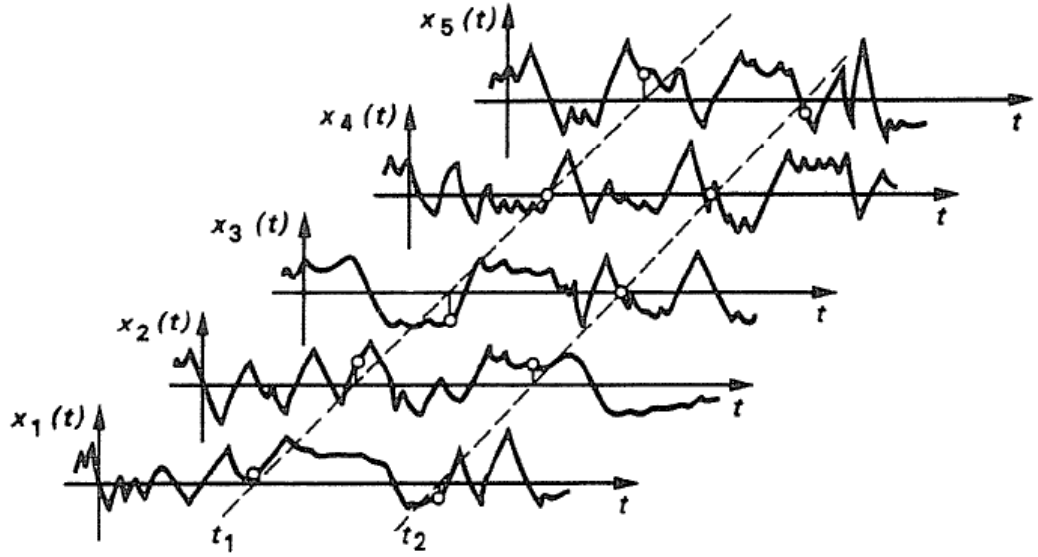


Fig. 2.4 Concept of ensemble averaging

Ensemble averaging

- Notice that if a process is ergodic, it must also be stationary, because an average along a single sample will (in theory) extend from t=-∞ to t =+∞ and will therefore be independent of time. If the sample and ensemble averages are the same, the ensemble averages must therefore be independent of absolute time, and so the process must be stationary.

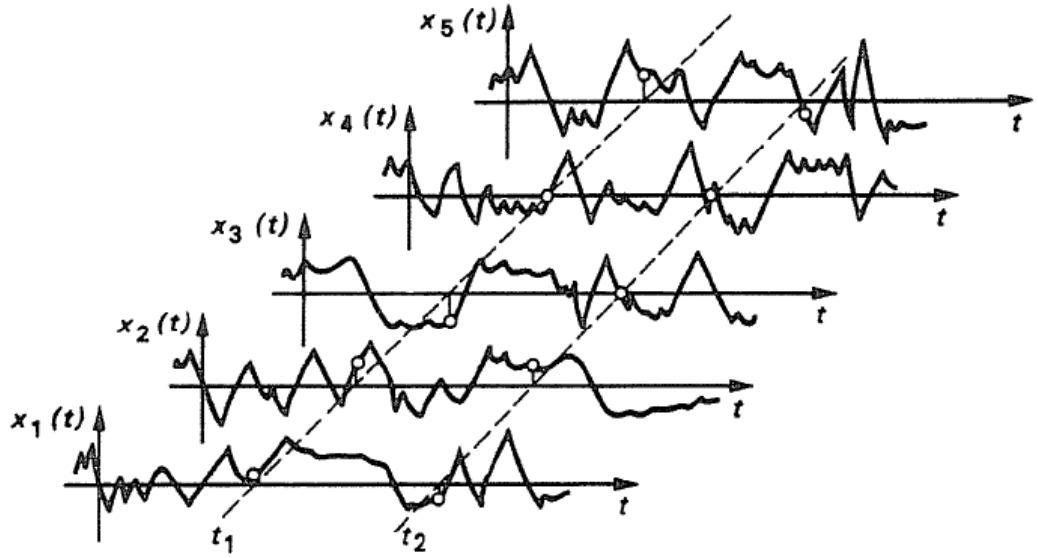
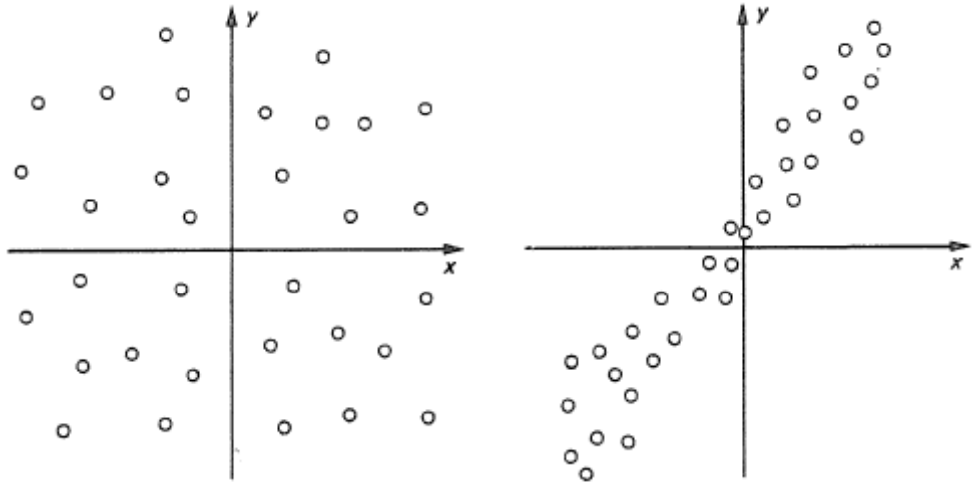


Fig. 2.4 Concept of ensemble averaging

Correlation

- Consider a population of pairs of two random variables x and y. Suppose that each pair of values is represented by a point on a graph of y against x.In the Figure a, the values of x and y in each pair have no apparent pattern whereas in Figure b, there is a definite pattern.
- For the latter case, in any pair, a large value of x is associated with a larger value of y, a small value of x with a small value of y and so on.

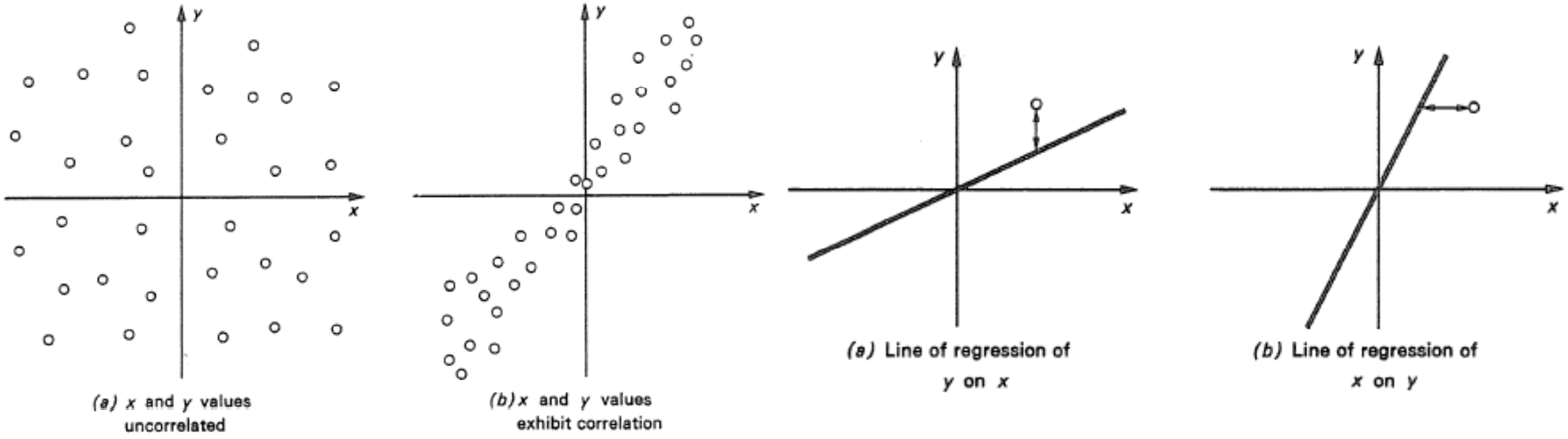


(a) x and y values uncorrelated

(b) x and y values exhibit correlation

Correlation

- The variables in Figure b are accordingly said to be correlated whereas those in Figure a are uncorrelated.
- In Figure b, we wish to express a functional relationship between x and y in the form of a straight line, one way of doing this is to minimize the square of the deviation of the actual values of y from their values predicted by the straight line approximation.



Correlation

- If the positions of the axes are adjusted so that

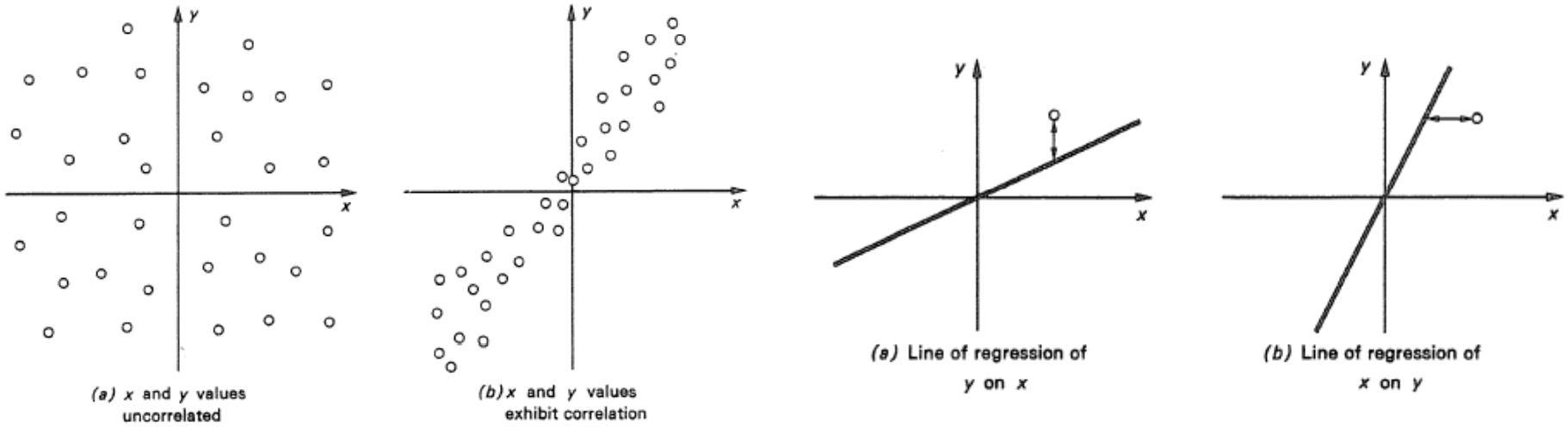
$$E[x] = E[y] = 0$$

i.e. So that the origin lies at the center of gravity of the data points, then the straight line approximation passes through the origin and can be written

$$y = mx.$$

- The deviation of any value of y from its predicted value mx is then

$$\Delta = y - mx$$



Correlation

- The average value of the square of the deviation is

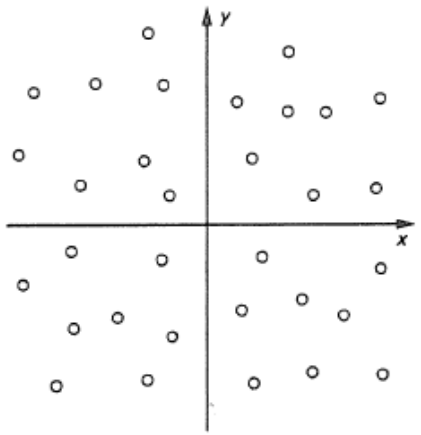
$$E[\Delta^2] = E[(y - mx)^2]$$
$$= E[y^2] + m^2E[x^2] - 2mE[xy]$$

which is a minimum when, by differentiating with respect to m,

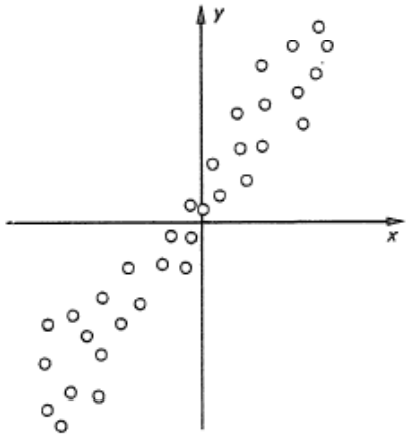
$$0 = 2mE[x^2] - 2E[xy]$$

or

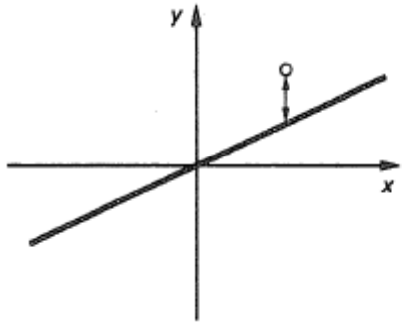
$$m = \frac{E[xy]}{E[x^2]}.$$



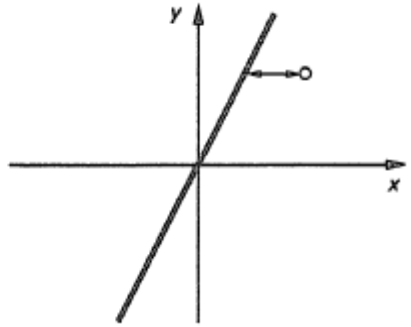
(a) x and y values uncorrelated



(b) x and y values exhibit correlation



(a) Line of regression of y on x



(b) Line of regression of x on y

Correlation

- Substituting this optimum value of slope m

$$m = \frac{E[xy]}{E[x^2]}.$$

into

$$y = mx.$$

gives:

$$y = \frac{E[xy]}{E[x^2]} \cdot x$$

- From the equation

$$\sigma^2 = E\left[x^2\right] - \left(E[x]\right)^2$$

or

$$\left(\text{variance}\right) = \left(\text{standard deviation}\right)^2 = \left\{\text{Mean square} - \left(\text{Mean}\right)^2\right\}$$

for zero mean

$$\sigma_x^2 = E[x^2] \qquad \text{and} \qquad \sigma_y^2 = E[y^2]$$

Correlation

- The equation

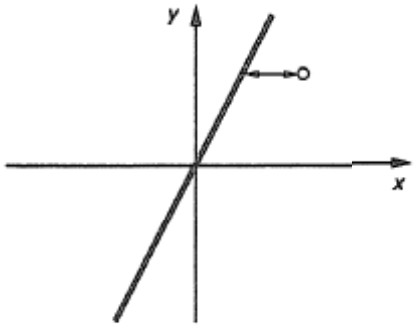
$$y = \frac{E[xy]}{E[x^2]} \cdot x$$

may be written finally as:

$$\frac{y}{\sigma_y} = \left\{\frac{E[xy]}{\sigma_x \sigma_y}\right\} \frac{x}{\sigma_x}.$$

- This is the equation for the line of regression of y on x. Alternatively, had we calculated the deviation of x from its predicted value shown in the figure, we should have obtained the line of regression of x on y given by:

$$\frac{x}{\sigma_x} = \left\{\frac{E[xy]}{\sigma_x \sigma_y}\right\} \frac{y}{\sigma_y}.$$



(b) Line of regression of *x* on *y*

Correlation

- The equation

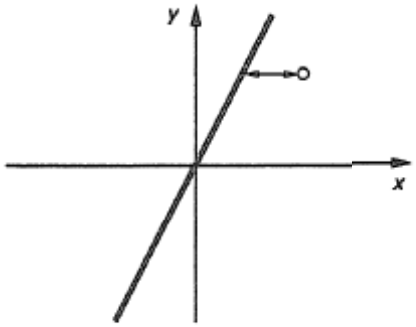
$$y = \frac{E[xy]}{E[x^2]} \cdot x$$

may be written finally as:

$$\frac{y}{\sigma_y} = \left\{\frac{E[xy]}{\sigma_x \sigma_y}\right\} \frac{x}{\sigma_x}.$$

- This is the equation for the line of regression of y on x. Alternatively, had we calculated the deviation of x from its predicted value shown in the figure, we should have obtained the line of regression of x on y given by:

$$\frac{x}{\sigma_x} = \left\{\frac{E[xy]}{\sigma_x \sigma_y}\right\} \frac{y}{\sigma_y}.$$



(b) Line of regression of *x* on *y*

Correlation

- In the case when x and y do not have zero means,as assumed above, the corresponding equations are:

$$\frac{y - m_y}{\sigma_y} = \left\{\frac{E[(x - m_x)(y - m_y)]}{\sigma_x \sigma_y}\right\} \frac{x - m_x}{\sigma_x}$$

and

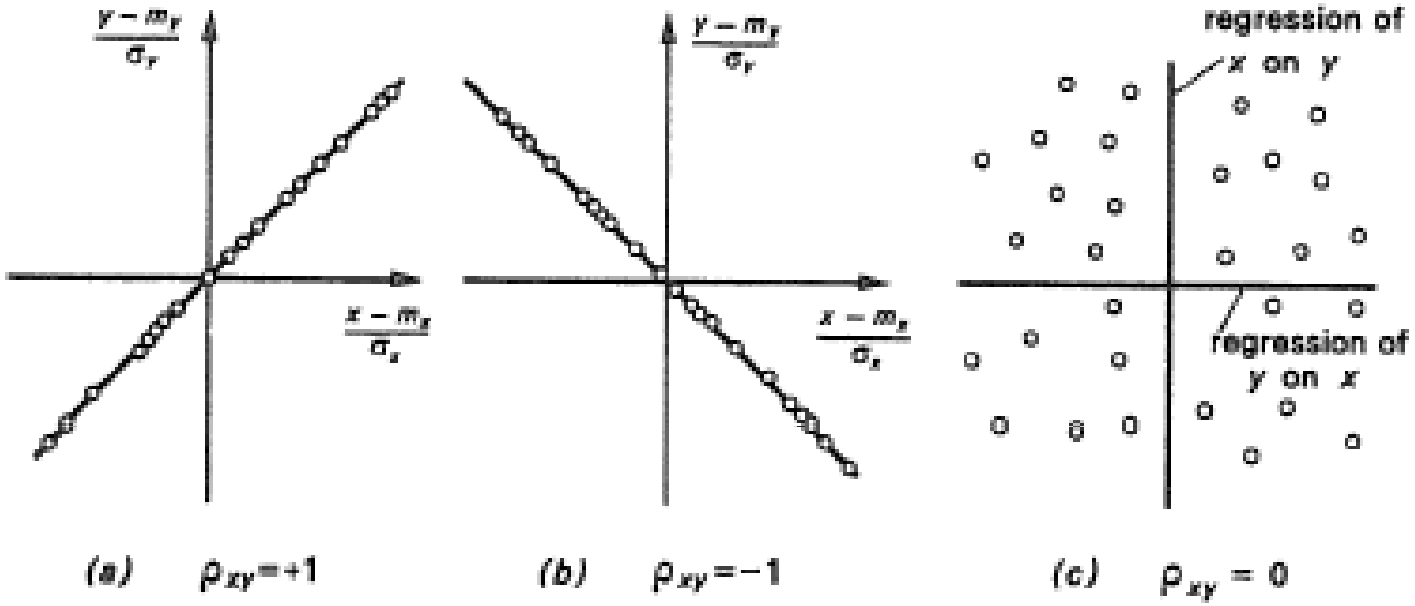
$$\frac{x - m_x}{\sigma_x} = \left\{\frac{E[(x - m_x)(y - m_y)]}{\sigma_x \sigma_y}\right\} \frac{y - m_y}{\sigma_y}$$

where mx and my are the mean values of x and y, respectively. The parameter

$$\rho_{xy} = \frac{E[(x - m_x)(y - m_y)]}{\sigma_x \sigma_y}$$

is called the correlation coefficient or normalized covariance and it will be clear that the first two equations above only represent the same straight line if $\rho_{xy} = \pm 1$, in which case there is a perfect correlation. In this case, only one random variable exists and one of the variables can be computed from the other variable as they are dependent to each other. If $\rho_{xy} = 0$, there is no correlation and the regression lines are parallel to the x and y axes, respectively. In this case, the random variables are completely indepedent of each other.

Correlation



Example

- Consider now two sine waves of constant amplitude and frequency and with a fixed phase difference ϕ so that

$$x(t) = x_0 \sin \omega t$$

$$y(t) = y_0 \sin (\omega t + \phi).$$

- Suppose that these two waves are sampled at an arbitrary time to, and we calculate the average value of the product $x(t_o)y(t_o)$ according to

$$E[f(x,y)] = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x,y)\, p(x,y)\, \mathrm{d}x\, \mathrm{d}y$$

- The average value is then given by:

$$E[x(t_0)y(t_0)] = \int_{-\infty}^{\infty} x_0 y_0 \sin \omega t_0 \sin (\omega t_0 + \phi) p(t_0)\, \mathrm{d}t_0$$

- The double integral reduces to a single integral since t_o is the only random variable.

Example

- In this case, we need to consider only t_o varying from 0 to $2\pi/\omega$ to cover a single full cycle of the periodic motion, so that the probability density function for t_o will be as shown in the figure.

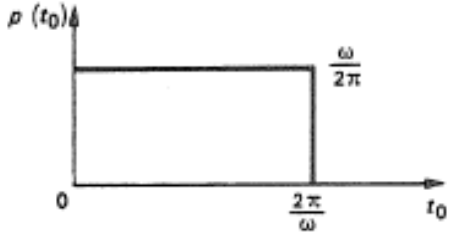


Fig. 3.4 Probability density function for the random time of sampling t_0

- Substituting this into

$$E[x(t_0)y(t_0)] = \int_{-\infty}^{\infty} x_0 y_0 \sin \omega t_0 \sin(\omega t_0 + \phi) p(t_0)\, dt_0$$

gives

$$\begin{aligned} E[x(t_0)y(t_0)] &= x_0 y_0 \left(\frac{\omega}{2\pi}\right) \int_0^{2\pi/\omega} \sin \omega t_0 \sin(\omega t_0 + \phi)\, dt_0 \\ &= x_0 y_0 \left(\frac{\omega}{2\pi}\right) \int_0^{2\pi/\omega} \{\sin^2 \omega t_0 \cos \phi + \sin \omega t_0 \cos \omega t_0 \sin \phi\}\, dt_0 \\ &= \tfrac{1}{2} x_0 y_0 \cos \phi \end{aligned}$$

Example

- Substituting $\sigma_x = \frac{x_o}{\sqrt{2}}$ and $\sigma_y = \frac{y_o}{\sqrt{2}}$

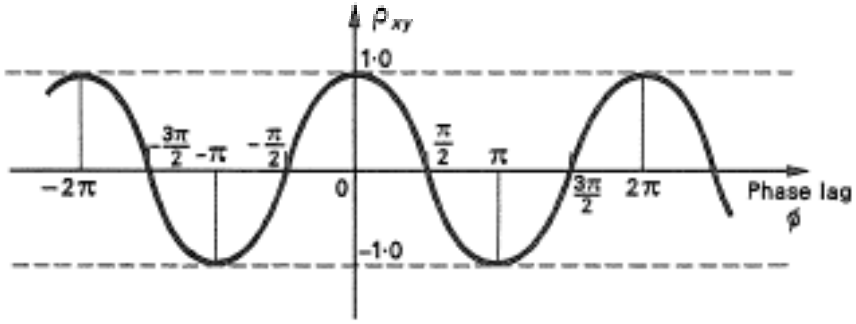
$$E[x(t_0)y(t_0)] = x_0 y_0 \left(\frac{\omega}{2\pi}\right) \int_0^{2\pi/\omega} \sin \omega t_0 \sin(\omega t_0 + \phi)\, dt_0$$

$$= x_0 y_0 \left(\frac{\omega}{2\pi}\right) \int_0^{2\pi/\omega} \{\sin^2 \omega t_0 \cos\phi + \sin \omega t_0 \cos \omega t_0 \sin\phi\}\, dt_0$$

$$= \tfrac{1}{2} x_0 y_0 \cos\phi$$

into $$\rho_{xy} = \frac{E[xy]}{\sigma_x \sigma_y}$$

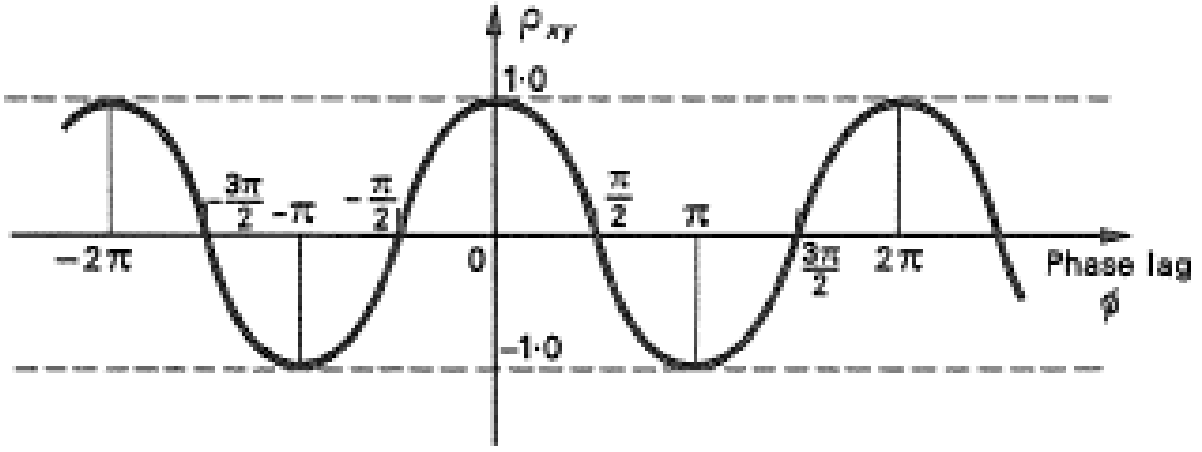
gives $$\rho_{xy} = \frac{E[xy]}{\sigma_x \sigma_y} = \cos\phi$$



Example

- The two sine waves may therefore said to be perfectly correlated when their phase difference is 0 or 180^o but uncorrelated when their phase difference is 90^o or 270^o .

- In general, two harmonic functions of time will be correlated if they move in phase or anti-phase, and uncorrelated if they are in quadrature to each other. This conclusion is important in understanding the form of autocorrelation function, to which we turn next.



Autocorrelation

- The autocorrelation function for a random process x(t) is defined as the average value of the product $x(t)x(t+\tau)$. The process is sampled at time t and then again at time $t+\tau$ and the average value of the product $E[x(t)x(t+\tau)]$, calculated for the ensemble.
- When mean values are subtracted from signals before computing an **autocorrelation** function, the resulting function is usually called an auto-**covariance** function.

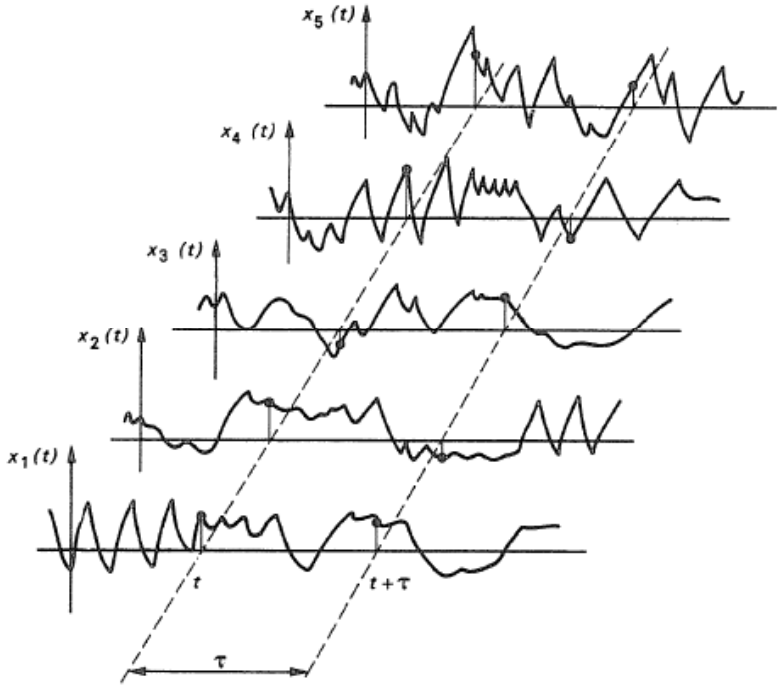


Fig. 3.6 Calculation of autocorrelation

Autocorrelation

- **Autocorrelation** is the **cross-correlation** of a signal with itself. It is a mathematical tool for finding repeating patterns, such as the presence of a periodic signal which has been buried under noise, or identifying the missing fundamental frequency in a signal implied by its harmonic frequencies.

- It is used frequently in signal processing for analyzing functions or series of values, such as time domain signals.

- Informally, it is the similarity between observations as a function of the time separation between them.

Autocorrelation

- Provided that the process is stationary, the value of $E[x(t)x(t+\tau)]$ will be independent of absolute time t and will depend only on the time separation τ so that we may put

$$E[x(t)x(t+\tau)] = f(\tau) = R_x(\tau)$$

where $R_x(\tau)$ is the autocorrelation function for x(t).

- We can deduce at once some of the properties of $R_x(\tau)$. Firstly, if x(t) is stationary, the mean and the standard deviation will be independent of t, so that

$$E[x(t)] = E[x(t+\tau)] = m$$

and

$$\sigma_{x(t)} = \sigma_{x(t+\tau)} = \sigma.$$

Autocorrelation

- The correlation coefficient for x(t) and x(t+τ) defined by

$$\rho_{xy} = \frac{E[(x - m_x)(y - m_y)]}{\sigma_x \sigma_y}$$

is therefore given by:

$$\rho = \frac{E[\{x(t) - m\}\{x(t+\tau) - m\}]}{\sigma^2}$$

$$= \frac{E[x(t)x(t+\tau)] - mE[x(t+\tau)] - mE[x(t)] + m^2}{\sigma^2}$$

$$= \frac{R_x(\tau) - m^2}{\sigma^2}$$

- Hence $R_x(\tau) = \sigma^2\rho + m^2$, and since the limiting values of ρ are ± 1, it follows that

$$-\sigma^2 + m^2 \leqslant R_x(\tau) \leqslant \sigma^2 + m^2.$$

- The value of autocorrelation function can therefore never be greater than the mean square value $E[x^2] = \sigma^2 + m^2$ and it can never be less than $-\sigma^2+m^2$.

Autocorrelation

- When the time interval τ separating the two measuring points is zero then

$$R_x(\tau = 0) = E[x(t)^2] = E[x^2]$$

and is just equal to the mean square value for the process.

- At very large time intervals, $\tau \to \infty$, a random process will be uncorrelated since there will not be a coherent relationship between the two values x(t) and x (t+τ) and so the correlation coefficient $\rho \to 0$. In this case, from

$$\rho = \frac{R_x(\tau) - m^2}{\sigma^2}$$

we get

$$R_x(\tau \to \infty) \to m^2.$$

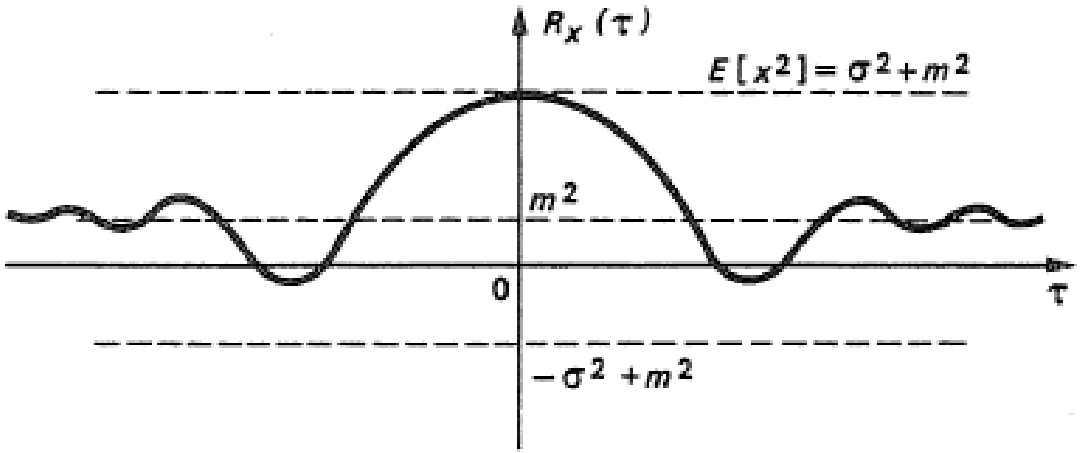
- Finally, since for a stationary process, $R_x(\tau)$ depends only on the separation time τ and not on absolute time t,

$$R_x(\tau) = E[x(t)x(t + \tau)] = E[x(t)x(t - \tau)] = R_x(-\tau)$$

so that $R_x(\tau)$ is an even function of τ. This result is obvious due to the assumed stationarity of the process.

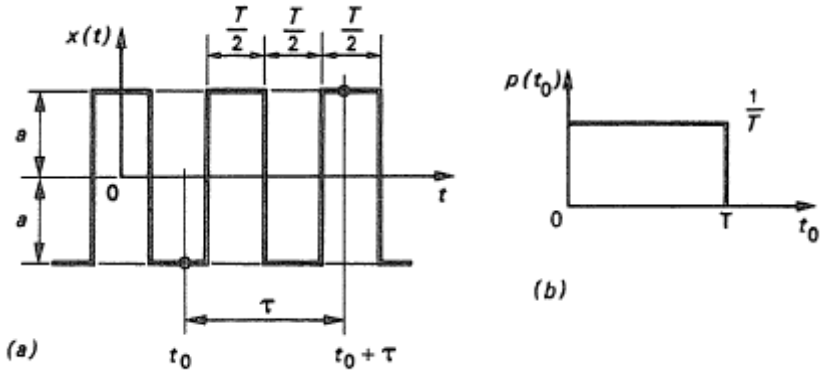
Autocorrelation

- All these properties are illustrated in a typical graph of an autocorrelation function $R_x(\tau)$ against separation time τ shown in the figure.



Example

- Calculate the autocorrelation function for an ergodic random process x(t) each of whose sample functions is a square wave of amplitude a and period T, but whose phase (that is the time of first switching after t=0) is a random variable uniformly distributed between 0 and T.



- There are two ways of making this calculation. The first is to calculate an ensemble average, by looking across the ensemble of sample functions at two fixed times t_o and $t_o + \tau$.

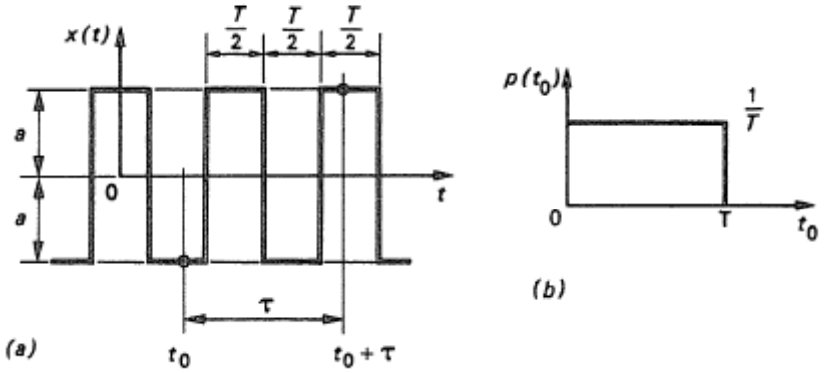
Example

- The second way makes use of the fact that the process is ergodic, when, as we have seen, any one sample function is completely representative of the process as a whole. The method is to average along a sample function, by thinking of the measuring time to as a random variable uniformly distributed along the time axis. In this case from

$$E[f(x,y)] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\,p(x,y)\,\mathrm{d}x\,\mathrm{d}y$$

we obtain:

$$R_x(\tau) = E[x(t_0)x(t_0+\tau)] = \int_{-\infty}^{\infty} x(t_0)x(t_0+\tau)p(t_0)\,\mathrm{d}t_0.$$

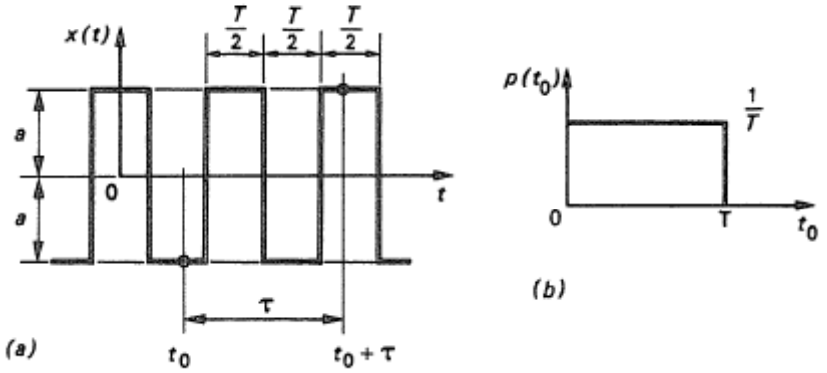


Example

- Since x(t) is periodic, we need only consider a single full cycle of the time history, and the range of values of t_o need only extend from 0 to T. All values within this range are equally likely, and therefore the probability distribution $p(t_o)$ for to is as shown in Figure b. Substituting this function into the above integral for $R_x(\tau)$ gives

$$R_x(\tau) = \int_0^T x(t_0)x(t_0 + \tau)\,\frac{1}{T}\,dt_0.$$

- Because x(t) is a discontinuous function, we must now proceed in steps. First consider the case when $0 \leqslant \tau \leqslant T/2.$



Example

- The integral

$$R_x(\tau) = \int_0^T x(t_0)x(t_0 + \tau)\,\frac{1}{T}\,dt_0.$$

then gives, after adjusting the time scale so that t_0=0 occurs at a switching point,

$$\begin{aligned} R_x(\tau) &= \frac{1}{T}\int_0^{T/2-\tau} a^2\,dt_0 + \frac{1}{T}\int_{T/2-\tau}^{T/2} -a^2\,dt_0 + \frac{1}{T}\int_{T/2}^{T-\tau} a^2\,dt_0 + \\ &\quad + \frac{1}{T}\int_{T-\tau}^{T} -a^2\,dt_0 \\ &= a^2\left(1 - 4\frac{\tau}{T}\right) \qquad \text{for} \qquad 0 \leqslant \tau \leqslant \frac{T}{2}. \end{aligned}$$

Next, let $T/2 \leqslant \tau \leqslant T$. In this case the integral in (3.19) gives

$$\begin{aligned} R_x(\tau) &= \frac{1}{T}\int_0^{T-\tau} -a^2\,dt_0 + \frac{1}{T}\int_{T-\tau}^{T/2} a^2\,dt_0 + \frac{1}{T}\int_{T/2}^{3T/2-\tau} -a^2\,dt_0 + \\ &\quad + \frac{1}{T}\int_{3T/2-\tau}^{T} a^2\,dt_0 \\ &= a^2\left(-3 + 4\frac{\tau}{T}\right) \qquad \text{for} \qquad \frac{T}{2} \leqslant \tau \leqslant T. \end{aligned}$$

Example

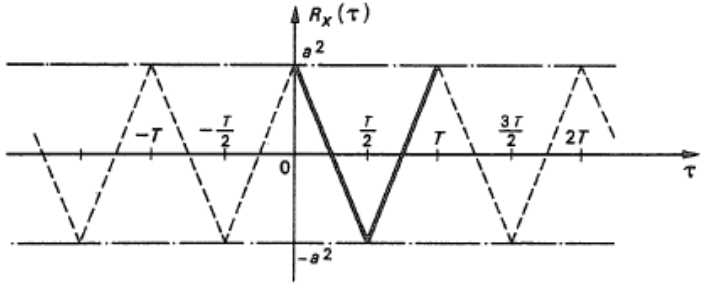
- If we plot these results, they appear as a triangular wave as shown in the figure. By continuing the process of integration for succesive stages, the dotted sections can easily be obtained, and the autocorrelation function for a square wave is therefore a triangular wave of constant amplitude.
- Notice that if τ=T,2T,3T, etc, the sample values x(to) and x(t_o+ τ) are always in phase with each other, so that there is a perfect correlation and

$$R_x(\tau) = E[x^2] = a^2$$

- If τ=T/2,3T/2,5T/2, etc, the sample values are always in antiphase and so

$$R_x(\tau) = -E[x^2] = -a^2$$

with again perfect correlation. However for intermediate values of τ, the samples are sometimes in phase and so there is then incomplete correlation.



Cross-correlation

- The cross-correlation functions between two different stationary random functions of time x(t) and y(t) are defined as:

$$R_{xy}(\tau) = E[x(t)y(t+\tau)]$$
$$R_{yx}(\tau) = E[y(t)x(t+\tau)].$$

- Because the processes are stationary it follows that

$$R_{xy}(\tau) = E[x(t-\tau)y(t)] = R_{yx}(-\tau)$$
$$R_{yx}(\tau) = E[y(t-\tau)x(t)] = R_{xy}(-\tau)$$

but in general, $R_{xy}(\tau)$ and $R_{yx}(\tau)$ are not the same and unlike the autocorrelation function, they are not even in τ.

- From

$$\rho_{xy} = \frac{E[(x-m_x)(y-m_y)]}{\sigma_x \sigma_y}$$

Cross-correlation

- Each cross-correlation function can be expressed in terms of the corresponding normalized covariance ρ to give:

$$R_{xy}(\tau) = \sigma_x \sigma_y \rho_{xy}(\tau) + m_x m_y$$

$$R_{yx}(\tau) = \sigma_y \sigma_x \rho_{yx}(\tau) + m_y m_x$$

and since the limiting value of the ρ's are ± 1 for perfect in-phase or anti-phase correlation, the limiting values of the cross-correlation functions must be

$$\pm \sigma_x \sigma_y + m_x m_y$$

so that for instance,

$$-\sigma_x \sigma_y + m_x m_y \leqslant R_{xy}(\tau) \leqslant \sigma_x \sigma_y + m_x m_y.$$

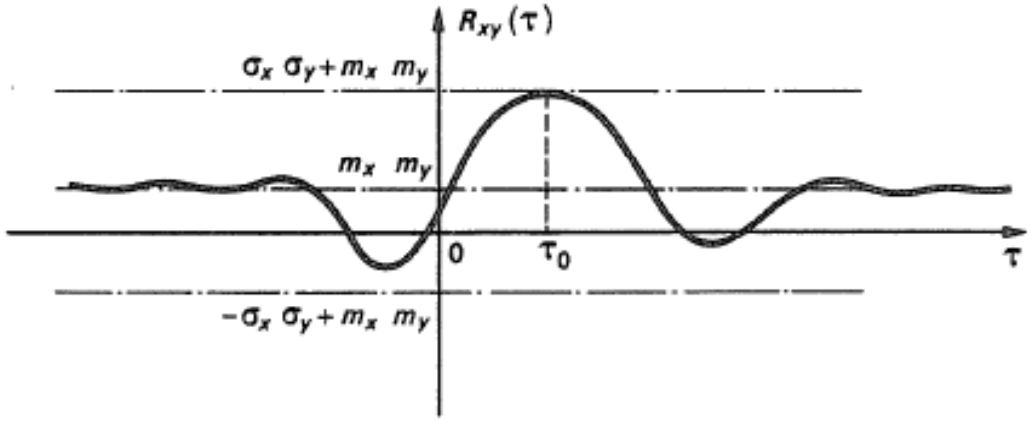
- For most random processes we expect that there will be no correlation between x and y when the time separation τ is very large, and therefore from the above two equations

$$R_{xy}(\tau \to \infty) \to m_x m_y$$

$$R_{yx}(\tau \to \infty) \to m_y m_x.$$

Cross-correlation

- These properties are illustrated in the figure which shows the possible form of a graph of cross-correlation function $R_{xy}(\tau)$ against separation time τ. In this case, it can be seen that the two random processes x(t) and y(t) show maximum correlation when $\tau=\tau_o$. It is shown in the next example that the phase of y(t) will be lagging that of x(t).



Example

- Consider two random processes x(t) and y(t) each of which consists of an ensemble of sample functions which are sine waves of the same constant amplitude and frequency. A typical sample of the x(t) process is given by:

$$x(t) = x_0 \sin(\omega t + \theta)$$

where θ is a constant phase angle. If θ were the same for all samples, x(t) would not be a random process, but, as in the previous example, we will assume that the phase varies from one sample to the next and is itself a random variable. If we assume that all phase angles between 0 and 2π are equally likely for each new sample function, then we can write:

$$p(\theta) = \begin{cases} 1/2\pi & \text{for} \quad 0 \leqslant \theta \leqslant 2\pi \\ 0 & \text{elsewhere} \end{cases}$$

- Next, suppose that although y(t) is a random process, each sample is related to a corresponding sample of the x(t) process so that corresponding to the sample for which $x(t) = x_0 \sin(\omega t + \theta)$, the corresponding sample of the y(t) process is $y(t) = y_0 \sin(\omega t + \theta - \phi)$ where ϕ is another constant phase angle which in this case we shall take to be the same for all samples. This could be the case if y(t) is derived from the x(t) process by an electrical or mechanical system which introduces a phase lag in its response.

Example

- The cross-correlation function $R_{xy}(\tau)$ can now be calculated by finding the ensemble average and obtaining

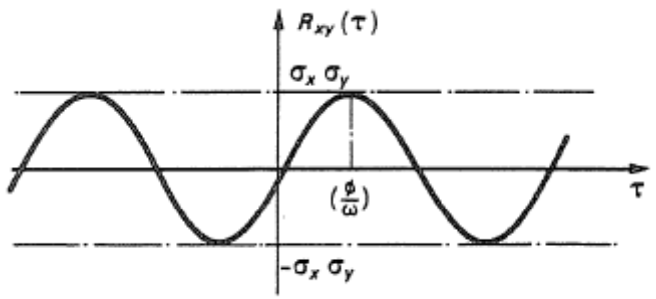
$$R_{xy}(\tau) = E[x(t)y(t+\tau)] = E[x_0 y_0 \sin(\omega t + \theta)\sin(\omega t + \omega\tau + \theta - \phi)]$$
$$= \int_0^{2\pi} x_0 y_0 \sin(\omega t + \theta)\sin(\omega t + \omega\tau + \theta - \phi)\frac{1}{2\pi}\,d\theta$$

which putting $\sin(\omega t + \omega\tau + \theta - \phi) = \sin(\omega t + \theta)\cos(\omega\tau - \phi) + \cos(\omega t + \theta)\sin(\omega\tau - \phi)$

and evaluating the integrals gives

$$R_{xy}(\tau) = \tfrac{1}{2}x_0 y_0 \cos(\omega\tau - \phi)$$

which is plotted in the figure.



Example

- The other cross-correlation function $R_{yx}(\tau)$ can be calculated in the same way, starting from

$$R_{yx}(\tau) = E[y(t)x(t+\tau)] = E[x_0 y_0 \sin(\omega t + \theta - \phi)\sin(\omega t + \omega\tau + \theta)]$$

and this leads to the result:

$$R_{yx}(\tau) = \tfrac{1}{2}x_0 y_0 \cos(\omega\tau + \phi)$$

in agreement with

$$R_{xy}(\tau) = E[x(t-\tau)y(t)] = R_{yx}(-\tau)$$

$$R_{yx}(\tau) = E[y(t-\tau)x(t)] = R_{xy}(-\tau)$$

- Since we have assumed that θ is uniformly distributed between 0 and 2π, the process x(t) and y(t) are both ergodic (notice that if θ was not so distributed, the ensemble average would depend on absolute time – consider the case when θ is always zero). We therefore have the option of calculating the sample averages rather than ensemble averages. If we pursue this alternative approach, we can proceed as in the previous example by assuming that typical sample functions x(t) and y(t) are themselves sampled at an arbitrary time to which may lie anywhere along the time axis.

Example

- It is sufficient to let to run from 0 to $2\pi/\omega$ (corresponding to one complete cycle) and we can imagine that to is itself a random variable with distribution

$$p(t_0) = \begin{cases} \omega/2\pi \quad \text{for} \quad 0 \leqslant t_0 \leqslant 2\pi/\omega \\ 0 \text{ elsewhere.} \end{cases}$$

- Then to find the sample average we have to integrate over to and obtain

$$R_{xy}(\tau) = \int_0^{2\pi/\omega} x_0 y_0 \sin(\omega t_0 + \theta)\sin(\omega t_0 + \omega\tau + \theta - \phi)\frac{\omega}{2\pi}\mathrm{d}t_0$$

which leads to the same result as before.

Covariance

- The covariance is defined as:

$$\mu_{xy} = E\left[\left(x - E(x)\right)\left(y - E(y)\right)\right] = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\left(x - E(x)\right)\left(y - E(y)\right)p(x,y)dxdy = E\left(xy\right) - E(x)E(y)$$

- It should be noted that when x and y are independent:

$$E\left(xy\right) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xyp(x)\,p(y)dxdy = E(x)E(y)$$

- In this case both the covariance and the correlation coefficient will be equal to zero.

Spectral density

- We now turn to the frequency composition of a naturally ocurring random process. Because the time history x(t) of a sample function is not periodic, it can not be represented by a discrete Fourier series. Also, for a stationary process, x(t) goes on forever and the condition

$$\int_{-\infty}^{\infty} |x(t)|\,dt < \infty$$

is not satisfied, so that the classical theory of Fourier analysis cannot be applied to a sample function. This difficulty can be overcome by analysing, not sample functions of the process itself, but its autocorrelation function $R_x(\tau)$.

- The logic behind this approach is that the autocorrelation function gives information about the frequencies present in a random process indirectly. We saw in the previous examples that $R_x(\tau)$ was a maximum for values of τ for which x(t) and x(t+ τ) were in phase and a minimum for values of τ for which they were in antiphase.

Spectral density

- The frequencies present in a graph of $R_x(\tau)$ against τ therefore reflect the frequency content of sample functions of the random process x(t).

- If the zero value of the random process x(t) is normalized (or adjusted) so that the mean value of the process m=E[x] is zero, then, provided that x(t) has no periodic components, $R_x(\tau \to \infty) = 0$ and the condition $\int_{-\infty}^{\infty} |R_x(\tau)|\, d\tau < \infty$ is satisfied. We can therefore apply the Fourier transform methods to calculate the Fourier transform of $R_x(\tau)$. The Fourier transform of $R_x(\tau)$ and its inverse transform are given by:

$$S_x(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} R_x(\tau)\, e^{-i\omega\tau}\, d\tau \qquad R_x(\tau) = \int_{-\infty}^{\infty} S_x(\omega)\, e^{i\omega\tau}\, d\omega$$

where $S_x(\omega)$ is called the spectral density of the x process and is a function of angular frequency ω.

Spectral density

- The most important property of $S_x(\omega)$ becomes apparent when we put $\tau=0$ in equation $R_x(\tau) = \int_{-\infty}^{\infty} S_x(\omega)\, e^{i\omega\tau}\, d\omega$ which from the fundamental definition of $R_x(\tau)$ in

$$E[x(t)x(t+\tau)] = f(\tau) = R_x(\tau)$$

gives:

$$E[x^2] = \int_{-\infty}^{\infty} S_x(\omega)\, d\omega.$$

- The mean square value of a stationary random process x is therefore given by the area under a graph of spectral density $S_x(\omega)$ against ω in the figure. The units of $S_x(\omega)$ are accordingly those of (mean square)/(unit of frequency) and a more complete name for $S_x(\omega)$ is the mean square spectral density or autospectral density.

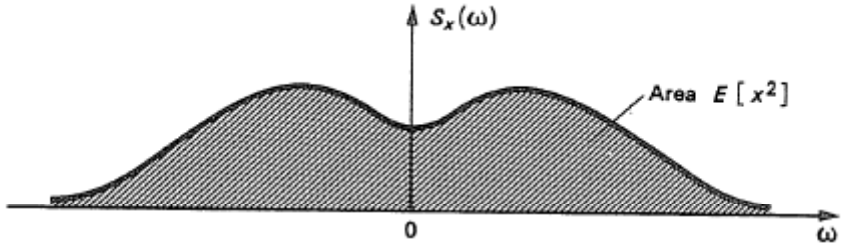


Fig. 5.1 The area under a spectral density curve is equal to $E[x^2]$

Spectral density

- The complex Fourier transform can be expressed in terms of its real and imaginary parts by

$$S_x(\omega) = A(\omega) - iB(\omega)$$

- Where from

$$A(\omega) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} x(t)\cos\omega t\, dt$$

$$B(\omega) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} x(t)\sin\omega t\, dt$$

- We get

$$A(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} R_x(\tau)\cos\omega\tau\, d\tau$$

$$B(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} R_x(\tau)\sin\omega\tau\, d\tau.$$

Spectral density

- Since $R_x(\tau)$ is an even function of τ while $\sin\omega\tau$ is an odd function, the product $R_x(\tau)\sin\omega\tau$ is an odd function and so the integral from $-\infty$ to 0 is exactly equal and opposite to the integration from 0 to ∞ in the below equation:

$$B(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} R_x(\tau)\sin\omega\tau\, d\tau.$$

- $B(\omega)$ is therefore zero and

$$S_x(\omega) = A(\omega)$$

- Which from

$$A(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} R_x(\tau)\cos\omega\tau\, d\tau$$

is a real even function of ω

Spectral density

- In addition, it can be shown that $S_x(\omega)$ is never negative, a fact which is supported by considering

$$E[x^2] = \int_{-\infty}^{\infty} S_x(\omega)\, d\omega.$$

- We would not expect the addition of frequency components $S_x(\omega)d\omega$ to reduce the mean square value $E[x^2]$. In summary therefore, the mean square spectral density of a stationary random process x(t) is real, even and nonnegative function of ω of the form illustrated in the figure.

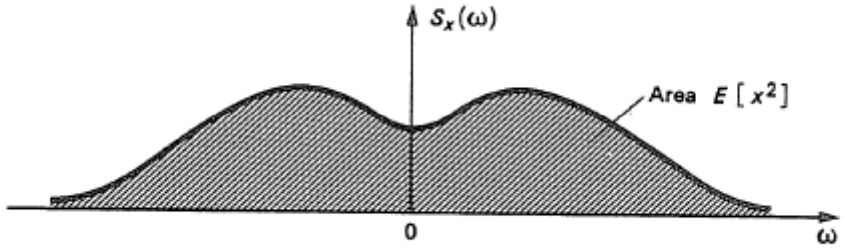


Fig. 5.1 The area under a spectral density curve is equal to $E[x^2]$

Example

- Determine the mean square and autocorrelation function for the stationary random process x(t) whose mean square spectral density is shown in the figure.

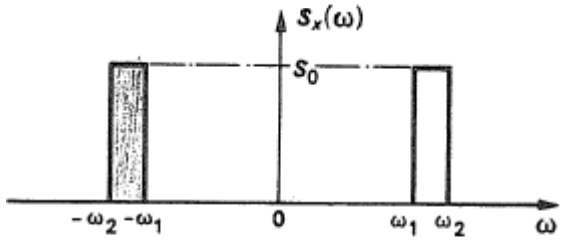


Fig. 5.2 Spectral density of a narrow band random process

- From

$$E[x^2] = \int_{-\infty}^{\infty} S_x(\omega)\,\mathrm{d}\omega.$$

- We get:

$$E[x^2] = \int_{-\infty}^{\infty} S_x(\omega)\,\mathrm{d}\omega = 2S_0(\omega_2 - \omega_1)$$

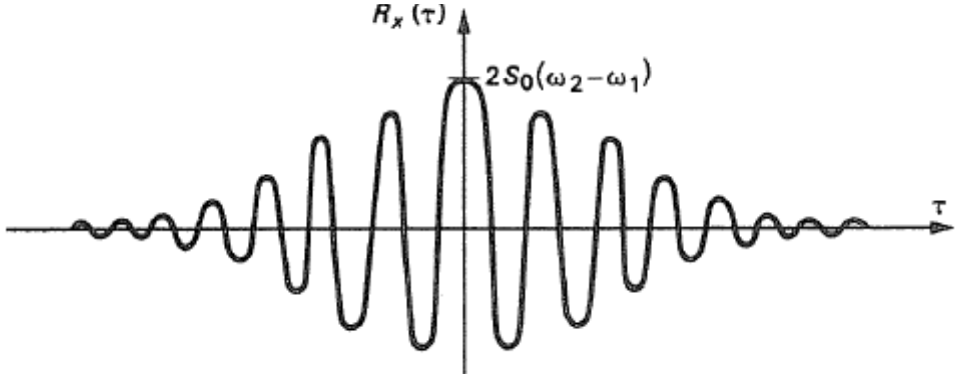
Example

- From

$$R_x(\tau) = \int_{-\infty}^{\infty} S_x(\omega)\, e^{i\omega\tau}\, d\omega$$
$$= \int_{-\infty}^{\infty} S_x(\omega) \cos\omega\tau\, d\omega$$

Since $S_x(\omega)$ is an even function of ω. Hence

$$R_x(\tau) = 2\int_{\omega_1}^{\omega_2} S_0 \cos\omega\tau\, d\omega$$
$$= 2S_0\left[\frac{1}{\tau}\sin\omega\tau\right]_{\omega_1}^{\omega_2}$$
$$= \frac{2S_0}{\tau}(\sin\omega_2\tau - \sin\omega_1\tau)$$
$$= \frac{4S_0}{\tau}\cos\left(\frac{\omega_1+\omega_2}{2}\right)\tau \cdot \sin\left(\frac{\omega_2-\omega_1}{2}\right)\tau$$

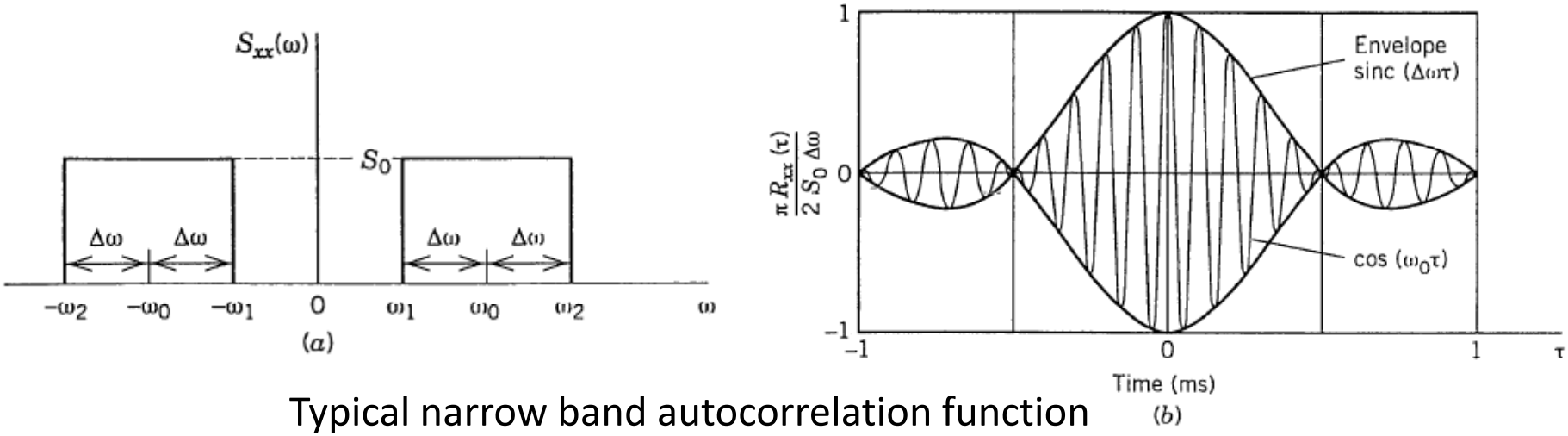


Autocorrelation function for a narrow band random process

Narrow band and broad band processes

- A narrow band random process is described as one where $\Delta\omega$ is small compared to ω_0.

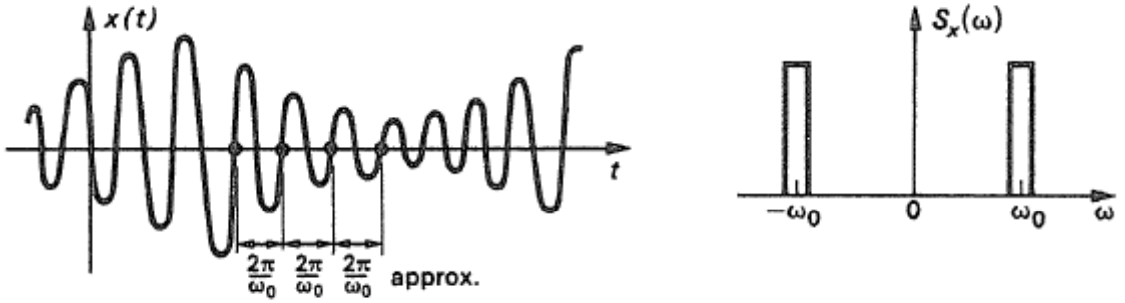
- Correlation is a maximum when τ=0 and follows a cosine graph of decreasing amplitude as correlation at the in-phase values of τ is gradually lost with increasing time separation τ



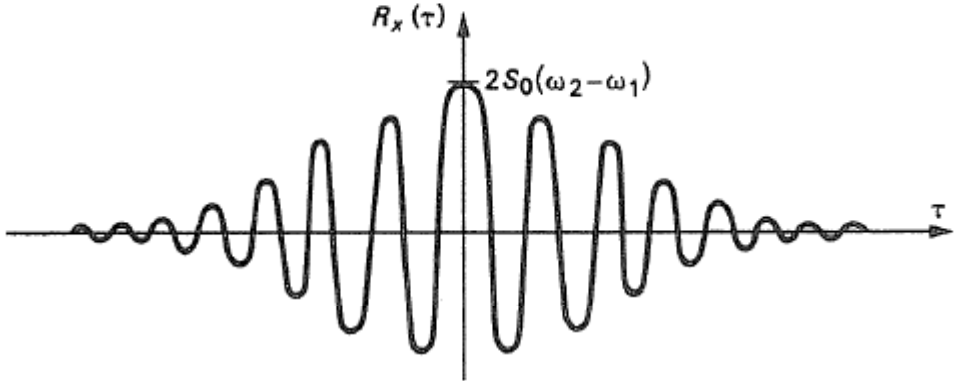
Typical narrow band autocorrelation function

Narrow band processes

- The narrower the bandwidth $\Delta\omega$, the larger τ must be before correlation is lost.



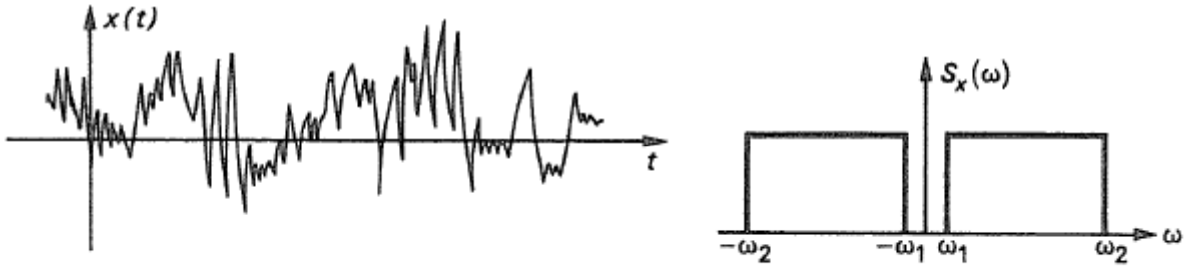
Time history of a sample from a narrow band process



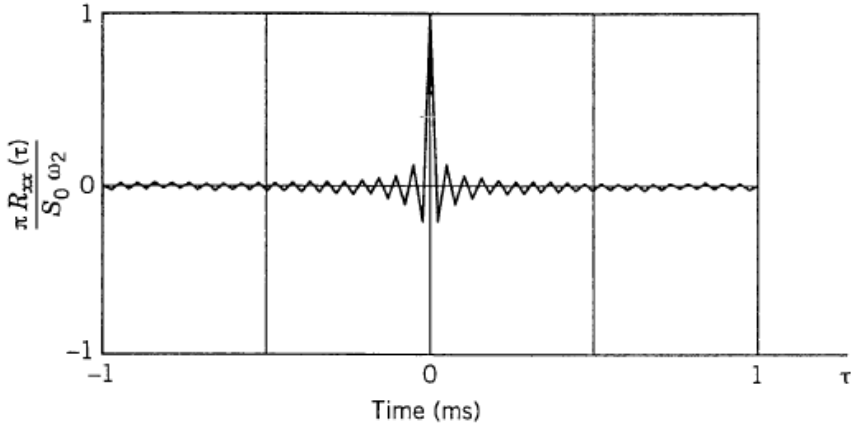
Autocorrelation function for a narrow band random process

Broad band processes

- A broad band process is one where whose spectral density covers a broad band of frequencies and the time history is then made up of the superposition of the whole band of frequencies as shown in the figure.



Time history of a sample from a broad band process



Typical broad-band auto-correlation function

White noise

- In the limit, when the frequency band extends from ω_1=0 to ω_2=∞, the spectrum is called white. From

$$E[x^2] = \int_{-\infty}^{\infty} S_x(\omega)\, d\omega.$$

the mean square value of a white noise process must be infinite, so white noise is only a theoretical concept, but in practical terms a spectrum is called white if it is broad band noise whose bandwidth extends well past all the frequencies of interest.

The "white" part of its name derives from the spectrum of white light, which contains the frequencies of all colors.

- The form of the autocorrelation function corresponding to white noise can be derived by considering the result of the above example. When the lower frequency limit ω_1=0, $R_x(\tau)$ becomes

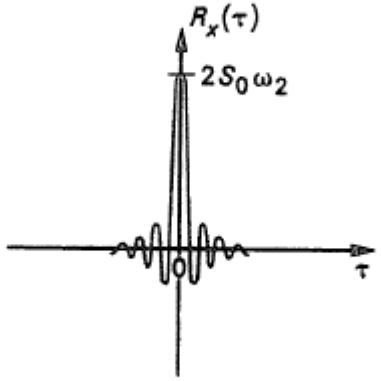
$$R_x(\tau) = \frac{4S_0}{\tau}\cos\frac{\omega_2\tau}{2}\sin\frac{\omega_2\tau}{2} = 2S_0\frac{\sin\omega_2\tau}{\tau}$$

White noise

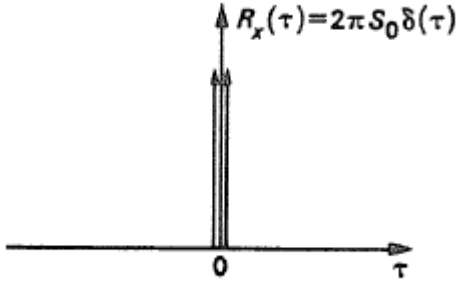
- The autocorrelation function for white noise

$$R_x(\tau) = \frac{4S_0}{\tau}\cos\frac{\omega_2\tau}{2}\sin\frac{\omega_2\tau}{2} = 2S_0\frac{\sin\omega_2\tau}{\tau}$$

is shown in the figure:



- When $\omega_2 \longrightarrow \infty$ adjacent cycles pack together so tightly that they become indistinguishable from a single vertical spike of infinite height, zero width, and finite area which we shall show has magnitude $2\pi S_0$
-



White noise

- This behaviour may be represented mathematically by using Dirac's delta function $\delta(\tau)$, which is an example of a generalized function.

- As mentioned already, delta (or impulse) functions play an important role in generalized Fourier analysis and allow the scope of the Fourier transforms to be greatly extended. The delta function is defined so that it is zero everywhere except at $\tau=0$, when it is infinite in such a way that

$$\int_{-\infty}^{\infty} \delta(\tau)\,\mathrm{d}\tau = 1.$$

- More generally, $\delta(\tau\text{-}T)$ is zero everywhere except at $\tau=T$, and has the property that

$$\int_{-\infty}^{\infty} \delta(\tau - T) f(\tau)\,\mathrm{d}\tau = f(\tau = T)$$

where $f(\tau)$ is any arbitrary continuous function of τ.

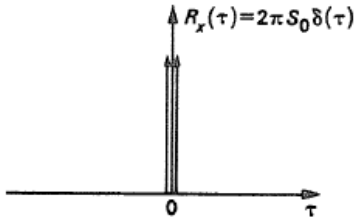
White noise

- Using this "delta function" notation, the autocorrelation function for a stationary random white noise process with spectral density S_o may be written

$$R_x(\tau) = 2\pi S_0\, \delta(\tau).$$

- The truth of this statement can be seen by noting that, according to the above equation $R_x(\tau)$ is zero everywhere except at $\tau=0$ where it is infinite, in agreement with the figure. The area under $R_x(\tau)$ at $\tau=0$ is $2\pi\, S_o$ and we can now verify that this is correct by calculating the Fourier transform of $R_x(\tau)$ to regain the spectral density $S_x(\omega)$, which, if all is well should come out to be the constant value S_o we originally specified. From $S_x(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} R_x(\tau)\, e^{-i\omega\tau}\, d\tau$

$$S_x(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} 2\pi S_0\, \delta(\tau)\, e^{-i\omega\tau}\, d\tau$$

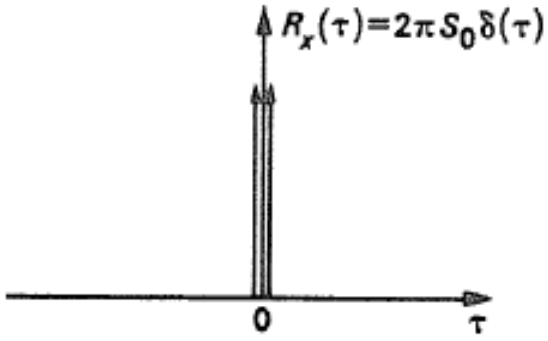


White noise

- Making use of $\int_{-\infty}^{\infty} \delta(\tau - T) f(\tau)\, d\tau = f(\tau = T)$ this does correctly reduce to:

$$S_x(\omega) = S_0.$$

- The autocorrelation function for white noise therefore involves a delta function at the origin, indicating zero correlation for all values of the separation time τ except at τ=0, where $R_x(0)=\infty$ since the mean square value of theoretical white noise is infinite.
- In $R_x(\tau) = 2\pi S_0\, \delta(\tau).$ the dimensions of So are those of (x^2)x(time) , those of $\delta(\tau)$ are from $\int_{-\infty}^{\infty} \delta(\tau - T) f(\tau)\, d\tau = f(\tau = T)$, 1/(time) , and so the dimensions of the autocorrelation function $R_x(\tau)$ are correctly those of x^2.



White noise

- White noise can be considered as completely random as x(t) is totally independent of x(t+τ) for all values of $\tau \neq 0$.

Spectral density of a derived process

- If we know the spectral density $S_x(\omega)$ of a stationary random process x(t), we can use this to calculate the mean square value $E[x^2]$ according to

$$E[x^2] = \int_{-\infty}^{\infty} S_x(\omega)\, d\omega.$$

- We can also use it to calculate the spectral density of processes which are obtained by differentiating x, for instance the velocity process

$$dx/dt = \dot{x} \text{ and } d^2x/dt^2 = \ddot{x}$$

- The calculation begins with the autocorrelation function

$$R_x(\tau) = E[x(t)x(t+\tau)]$$

- Which may be written for an ensemble average

$$R_x(\tau) = \lim_{N\to\infty} \frac{1}{N} \sum_{r=1}^{N} x_r(t)x_r(t+\tau).$$

- Consider differentiating $R_x(\tau)$ with respect to τ. In order to do so, we have to differentiate each term of the form $x_r(t)x_r(t+\tau)$ in the summation keeping t constant.

Spectral density of a derived process

- This gives:

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\{x_r(t)x_r(t+\tau)\} = x_r(t)\frac{\mathrm{d}}{\mathrm{d}\tau}x_r(t+\tau)$$

$$= x_r(t)\frac{\mathrm{d}}{\mathrm{d}(t+\tau)}x_r(t+\tau)\cdot\frac{\mathrm{d}(t+\tau)}{\mathrm{d}\tau}$$

$$= x_r(t)\dot{x}_r(t+\tau)$$

- And so we obtain

$$\frac{\mathrm{d}}{\mathrm{d}\tau}(R_x(\tau)) = E[x(t)\dot{x}(t+\tau)].$$

- For a stationary process, ensemble averages are independent of time t, so

$$E[x(t)\dot{x}(t+\tau)] = E[x(t-\tau)\dot{x}(t)]$$

- Giving

$$\frac{\mathrm{d}}{\mathrm{d}\tau}(R_x(\tau)) = E[x(t-\tau)\dot{x}(t)].$$

- Differentiating again wrt τ gives

$$\frac{\mathrm{d}^2}{\mathrm{d}\tau^2}(R_x(\tau)) = -E[\dot{x}(t-\tau)\dot{x}(t)]$$

$$= -R_{\dot{x}}(\tau)$$

where $R_{\dot{x}}(\tau)$ is the autocorrelation function for the derived process $\dot{x}(t)$.

Spectral density of a derived process

- Also from the Fourier integral $R_x(\tau) = \int_{-\infty}^{\infty} S_x(\omega)\, e^{i\omega\tau}\, d\omega$
- The right hand side of this equation is a definite integral wrt ω with τ held constant, and with the limits of integration independent of τ. It is therefore legitimate to differentiate wrt τ under the integral sign to obtain

$$\frac{d}{d\tau}(R_x(\tau)) = \int_{-\infty}^{\infty} i\omega\, S_x(\omega)\, e^{i\omega\tau}\, d\omega$$

$$\frac{d^2}{d\tau^2}(R_x(\tau)) = -\int_{-\infty}^{\infty} \omega^2\, S_x(\omega)\, e^{i\omega\tau}\, d\omega.$$

- Combining this with $\frac{d^2}{d\tau^2}(R_x(\tau)) = -E[\dot{x}(t-\tau)\dot{x}(t)]$

$$= -R_{\dot{x}}(\tau)$$

- We see that the autocorrelation function for the derived process can be expressed as:

$$R_{\dot{x}}(\tau) = \int_{-\infty}^{\infty} \omega^2\, S_x(\omega)\, e^{i\omega\tau}\, d\omega.$$

Spectral density of a derived process

- But, since $R_{\dot{x}}(\tau)$ can also be written as the inverse transform of the spectral density $S_{\dot{x}}(\omega)$ according to $R_x(\tau) = \int_{-\infty}^{\infty} S_x(\omega)\, e^{i\omega\tau}\, d\omega$. We also have: $R_{\dot{x}}(\tau) = \int_{-\infty}^{\infty} S_{\dot{x}}(\omega)\, e^{i\omega\tau}\, d\omega$
- By comparing the above equation with $R_{\dot{x}}(\tau) = \int_{-\infty}^{\infty} \omega^2 S_x(\omega)\, e^{i\omega\tau}\, d\omega.$
- It is clear that: $S_{\dot{x}}(\omega) = \omega^2 S_x(\omega),$
- So that the spectral density of the derived process is just ω^2 times the $S_x(\omega)$ spectral density of the original process. This is an important result because we can now calculate the mean square velocity $E[\dot{x}^2]$ from knowledge of since $E[\dot{x}^2] = \int_{-\infty}^{\infty} S_{\dot{x}}(\omega)\, d\omega = \int_{-\infty}^{\infty} \omega^2 S_x(\omega)\, d\omega.$
- Similarly, the mean square acceleration $E[\ddot{x}^2]$ is given by:

$$E[\ddot{x}^2] = \int_{-\infty}^{\infty} S_{\ddot{x}}(\omega)\, d\omega = \int_{-\infty}^{\infty} \omega^4 S_x(\omega)\, d\omega.$$

Cross- spectral density

- We have seen how the spectral density of a random process is defined as the Fourier transform of its autocorrelation function. In the same way, the cross-spectral density of a pair of random processes is defined as the Fourier transform of the corresponding cross-correlation function for the two processes. Therefore, if $R_{xy}(\tau)$ and $R_{yx}(\tau)$ are the two cross-correlation functions, we have:

$$S_{xy}(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} R_{xy}(\tau)\,\mathrm{e}^{-i\omega\tau}\,\mathrm{d}\tau$$

$$S_{yx}(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} R_{yx}(\tau)\,\mathrm{e}^{-i\omega\tau}\,\mathrm{d}\tau$$

- And their accompanying inverse transform relations which are:

$$R_{xy}(\tau) = \int_{-\infty}^{\infty} S_{xy}(\omega)\,\mathrm{e}^{i\omega\tau}\,\mathrm{d}\omega \qquad R_{yx}(\tau) = \int_{-\infty}^{\infty} S_{yx}(\omega)\,\mathrm{e}^{i\omega\tau}\,\mathrm{d}\omega.$$

Cross- spectral density

- According to classical Fourier transform theory we need:

$$\int_{-\infty}^{\infty} |R_{xy}(\tau)| \, d\tau < \infty$$

for the integrals to exist, and this means that x(t) and y(t+τ) must be uncorrelated when $\tau \to \infty$ and from

$$R_{xy}(\tau \to \infty) \to m_x m_y$$
$$R_{yx}(\tau \to \infty) \to m_y m_x.$$

that either mx or my (i.e. one of the mean values) must be zero. The cross-correlation functions are related by:

$$R_{xy}(\tau) = E[x(t-\tau)y(t)] = R_{yx}(-\tau)$$
$$R_{yx}(\tau) = E[y(t-\tau)x(t)] = R_{xy}(-\tau)$$

and it follows that the two cross-spectra also related.

Cross- spectral density

- Putting $R_{xy}(\tau) = R_{yx}(-\tau)$, the equation $S_{xy}(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} R_{xy}(\tau)\,e^{-i\omega\tau}\,d\tau$ gives:

$$S_{xy}(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} R_{yx}(-\tau)\,e^{-i\omega\tau}\,d\tau$$

- If we now substitute $\tau' = -\tau$, we obtain:

$$S_{xy}(\omega) = \frac{1}{2\pi}\int_{\tau'=\infty}^{\tau'=-\infty} R_{yx}(\tau')\,e^{i\omega\tau'}\,(-d\tau')$$

- Which is the same as $S_{yx}(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} R_{yx}(\tau)\,e^{-i\omega\tau}\,d\tau$ except that the sign of iω has been changed. If according to

$$e^{i\theta} = \cos\theta + i\sin\theta$$

$$X(\omega) = A(\omega) - iB(\omega)$$

$$S_{xy}(\omega) = A(\omega) - iB(\omega) \qquad S_{yx}(\omega) = C(\omega) - iD(\omega)$$

where A(ω), B(ω), C(ω) and D(ω) are real functions of ω, then by comparing

Cross- spectral density

- By comparing $S_{xy}(\omega) = \frac{1}{2\pi}\int_{\tau'=-\infty}^{\tau'=-\infty} R_{yx}(\tau')\,e^{i\omega\tau'}(-d\tau')$
- And $S_{yx}(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} R_{yx}(\tau)\,e^{-i\omega\tau}\,d\tau$
- We must have: $C(\omega) = A(\omega) \quad D(\omega) = -B(\omega)$
- Since the definite integrals are independent of the variables of integration, whether τ or τ'.
- Hence $S_{xy}(\omega)$ and $S_{yx}(\omega)$ are the same except that the sign of their imaginary parts is reversed. $S_{yx}(\omega)$ İs therefore the complex conjugate of $S_{xy}(\omega)$, which is usually written as:

$$S_{yx}(\omega) = S_{xy}^{*}(\omega) \qquad S_{xy}(\omega) = S_{yx}^{*}(\omega).$$

Example

- The white noise ergodic random process y(t) is the result of delaying a similar process x(t) by time T. (A sample of the x(t) process may be thought of as being tape recorded and then replayed T seconds later as y(t).) If the spectral density of x(t) and y(t) is S_o, determine the cross-correlation functions $R_{xy}(\tau)$ and $R_{yx}(\tau)$ and the cross-spectra $S_{xy}(\omega)$ and $S_{yx}(\omega)$. From

$$R_{xy}(\tau) = E[x(t)y(t+\tau)]$$
$$R_{yx}(\tau) = E[y(t)x(t+\tau)].$$

$$R_{xy}(\tau) = E[x(t)y(t+\tau)] = R_{yx}(-\tau)$$

- In this case,

$$y(t+T) = x(t)$$

- So that

$$y(t) = x(t-T)$$

- And

$$y(t+\tau) = x(t+\tau-T)$$

Example

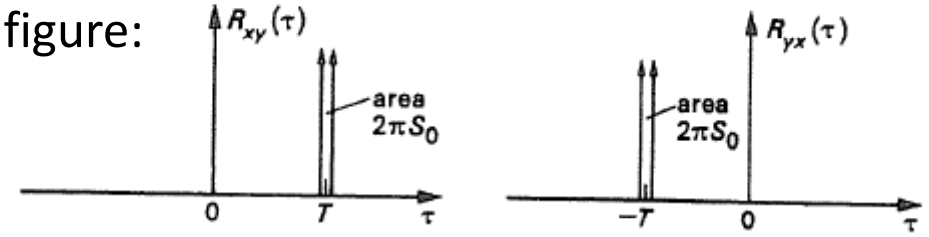
- Giving:

$$R_{xy}(\tau) = E[x(t)x(t + \tau - T)].$$

- The right hand side of this is just the autocorrelation function $R_x(\tau\text{-}T)$ which from $R_x(\tau) = 2\pi S_0\,\delta(\tau).$ is given by $2\pi S_0\,\delta(\tau - T),$ and so finally we have:

$$R_{xy}(\tau) = 2\pi S_0\,\delta(\tau - T) = R_{yx}(-\tau)$$

as shown in the figure:



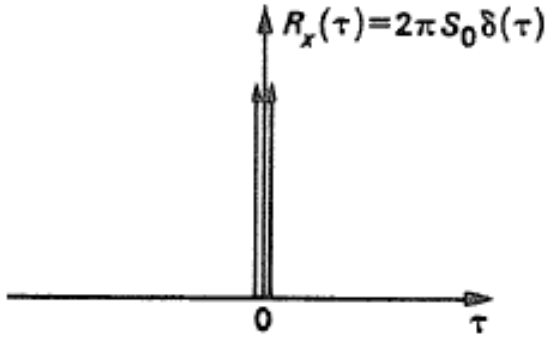
Cross-correlation functions for delayed white noise

- Since x(t) and y(t) (which is simply x(t) delayed by T) are both white noise processes, their cross-correlation functions are zero everywhere except for the single value of τ for which x=y. Then

$$R_{xy}(\tau = T) = R_{yx}(\tau = -T) = R_x(\tau = 0)$$

Example

- This is the delta function illustrated in the figure.



- From

$$S_{yx}(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} R_{yx}(\tau)\,e^{-i\omega\tau}\,d\tau$$

$$S_{yx}(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} 2\pi S_0\,\delta(\tau + T)\,e^{-i\omega\tau}\,d\tau$$
$$= S_0\,e^{i\omega T}$$

- Thus confirming

$$S_{yx}(\omega) = S_{xy}^*(\omega) \qquad S_{xy}(\omega) = S_{yx}^*(\omega).$$

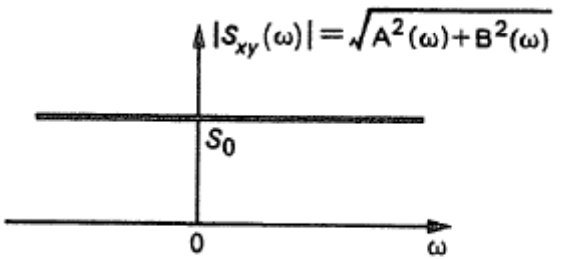
Example

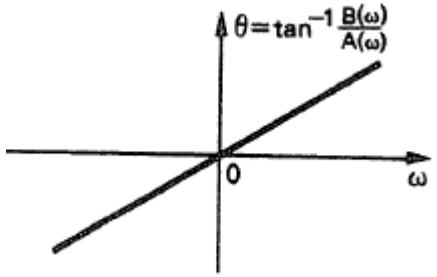
- Putting $S_{xy}(\omega) = A(\omega) - iB(\omega)$

$$A(\omega) = S_0 \cos \omega t$$

$$B(\omega) = S_0 \sin \omega t$$

- If we plot the magnitude of $|S_{xy}(\omega)|$ and $\theta = \tan^{-1} B(\omega)/A(\omega)$





Note on the units of spectral density

- We have already noted that the units of spectral density are those of x^2/(units of angular frequency) , while those of the cross-spectral density are the units of (x.y)/(units of angular frequency) . However, a potential pitfall occurs when it is necessary to ditinguish between :
  - A frequency scale from -∞ to +∞
  - A frequency scale from 0 to +∞
  - Frequencies expressed in Hz (c/s) instead of rad/s
- Perhaps the best way of avoiding any difficulty is to return to the basic formula

$$E[x^2] = \int_{-\infty}^{\infty} S_x(\omega)\,\mathrm{d}\omega$$

- And consider the practical form of this which is

$$E[x^2] = \int_{0}^{\infty} W_x(f)\,\mathrm{d}f$$

- Where f is frequency in Hz and $W_x(f)$ is the equivalent one-sided spectral density function

Note on the units of spectral density

- The frequency band ω to $\omega+d\omega$ rad/s in Fig a corresponds to $\omega/2\pi$ to $(\omega+d\omega)/ 2\pi$ Hz in Figure b, so that for equal contributions to the mean square in the frequency band, the shaded areas in both (a) and (b) must be the same. Hence

$$2S_x(\omega)\,d\omega = W_x\left(f = \frac{\omega}{2\pi}\right)\frac{d\omega}{2\pi}$$

- And so the single-sided spectrum $W_x(f)$ is related to the double-sided spectrum $Sx(\omega)$ by the formula

$$W_x\left(f = \frac{\omega}{2\pi}\right) = 4\pi S_x(\omega)$$

- Or

$$W_x(f) = 4\pi S_x(\omega = 2\pi f).$$

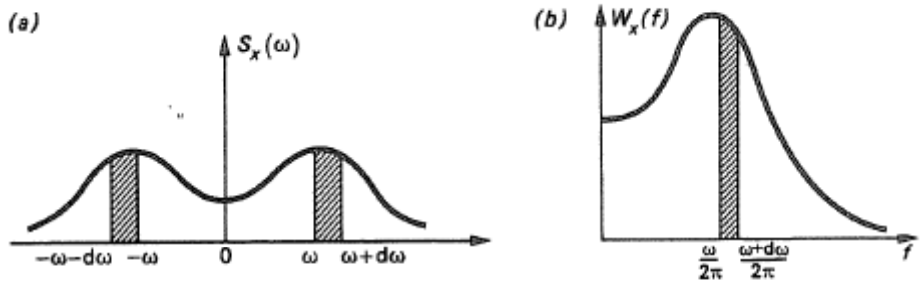


Fig. 5.9 Illustrating the relationship between alternative spectral density parameters