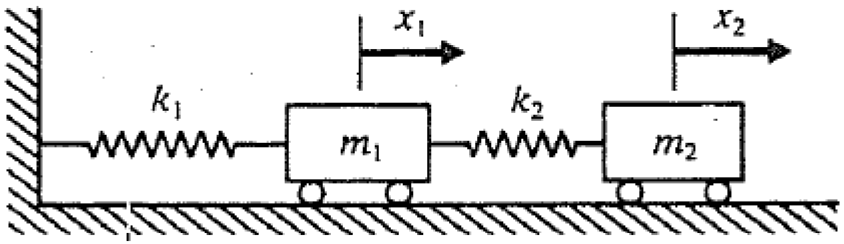
Two degree of freedom systems

- Equations of motion for forced vibration
- Free vibration analysis of an undamped system

Introduction

- Systems that require two independent coordinates to describe their motion are called two degree of freedom systems.

$$\begin{array}{c}\text{Number of}\\ \text{degrees of freedom}\\ \text{of the system}\end{array} = \begin{array}{c}\text{Number of masses}\\ \text{in the system}\end{array} \times \begin{array}{c}\text{number of possible types}\\ \text{of motion of each mass}\end{array}$$



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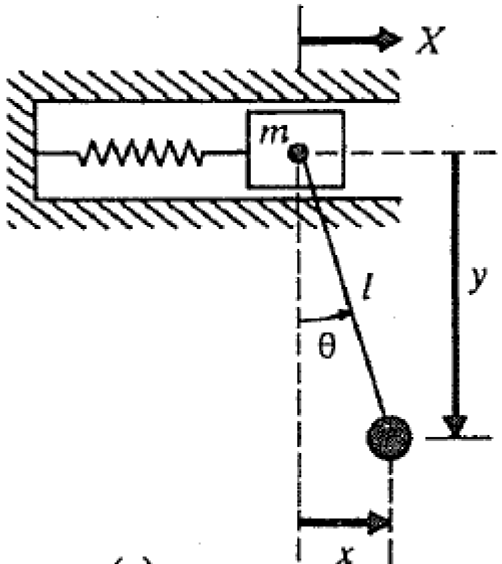

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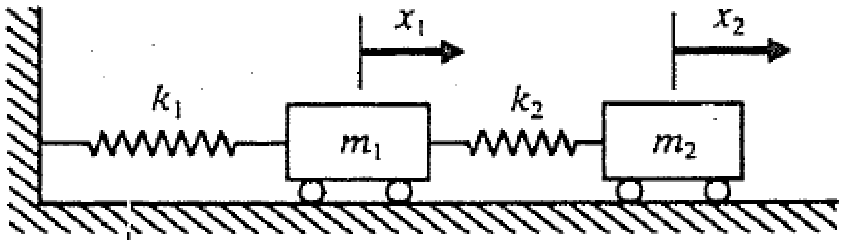
Introduction

- There are two equations for a two degree of freedom system, one for each mass (precisely one for each degree of freedom).
- They are generally in the form of coupled differential equations-that is, each equation involves all the coordinates.
- If a harmonic solution is assumed for each coordinate,the equations of motion lead to a frequency equation that gives two natural frequencies of the system.



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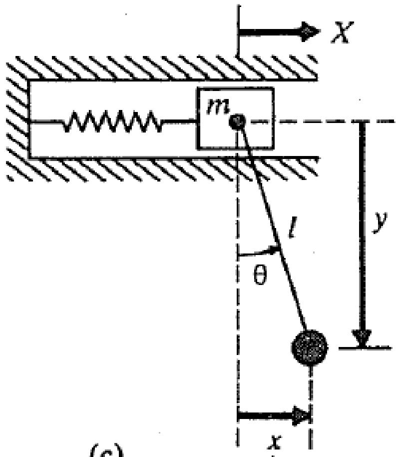

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Introduction

- If we give suitable initial excitation, the system vibrates at one of these natural frequencies. During free vibration at one of the natural frequencies, the amplitudes of the two degrees of freedom (coordinates) are related in a specified manner and the configuration is called a **normal mode**, **principle mode**, or **natural mode of vibration**.

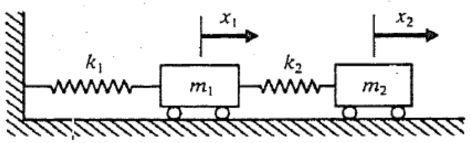
- Thus a **two degree of freedom** system has **two normal modes** of vibration corresponding to **two natural frequencies**.

- If we give an arbitrary initial excitation to the system, the resulting free vibration will be a superposition of the two normal modes of vibration. However, if the system vibrates under the action of an external harmonic force, the resulting forced harmonic vibration takes place at the frequency of the applied force.

Introduction

- As is evident from the systems shown in the figures, the configuration of a system can be specified by a set of independent coordinates such as length, angle or some other physical parameters. Any such set of coordinates is called **generalized coordinates**.

- Although the equations of motion of a two degree of freedom system are generally coupled so that each equation involves all coordinates, it is always possible to find a particular set of coordinates such that each equation of motion contains only one coordinate. The equations of motion are then **uncoupled** and can be solved independently of each other. Such a set of coordinates, which leads to an uncoupled system of equations, is called **principle copordinates**.



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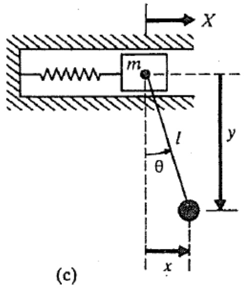

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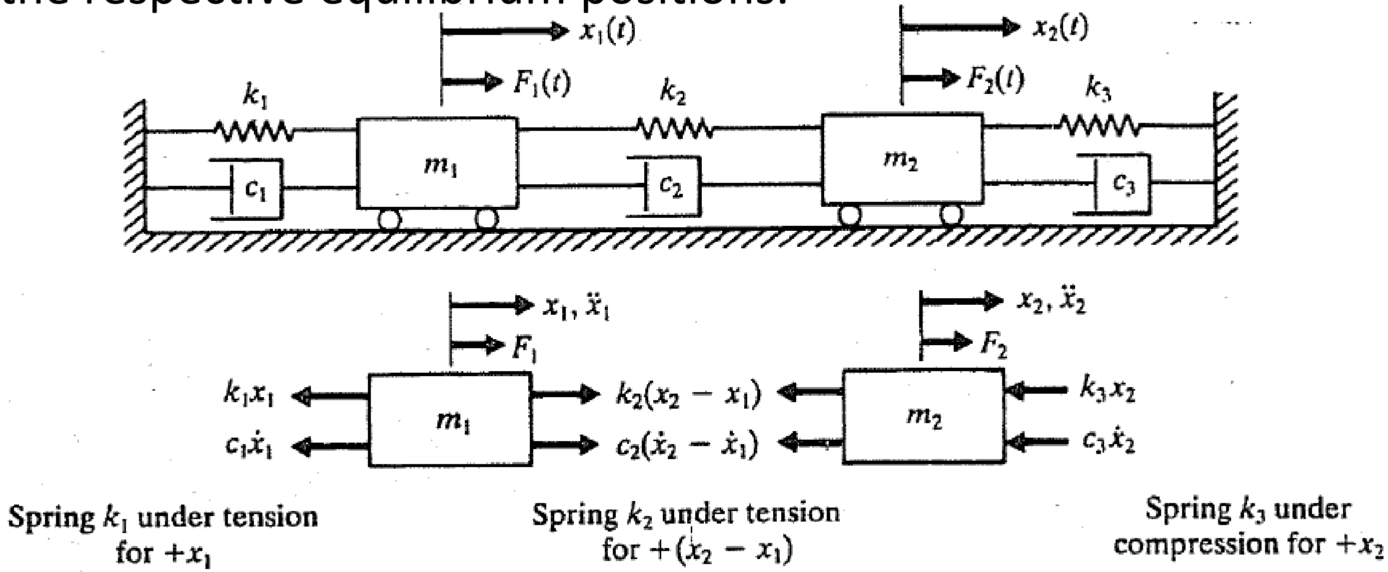
Equations of motion for forced vibration

- Consider a viscously damped two degree of freedom spring-mass system shown in the figure.

- The motion of the system is completely described by the coordinates $x_1(t)$ and $x_2(t)$, which define the positions of the masses m_1 and m_2 at any time t from the respective equilibrium positions.

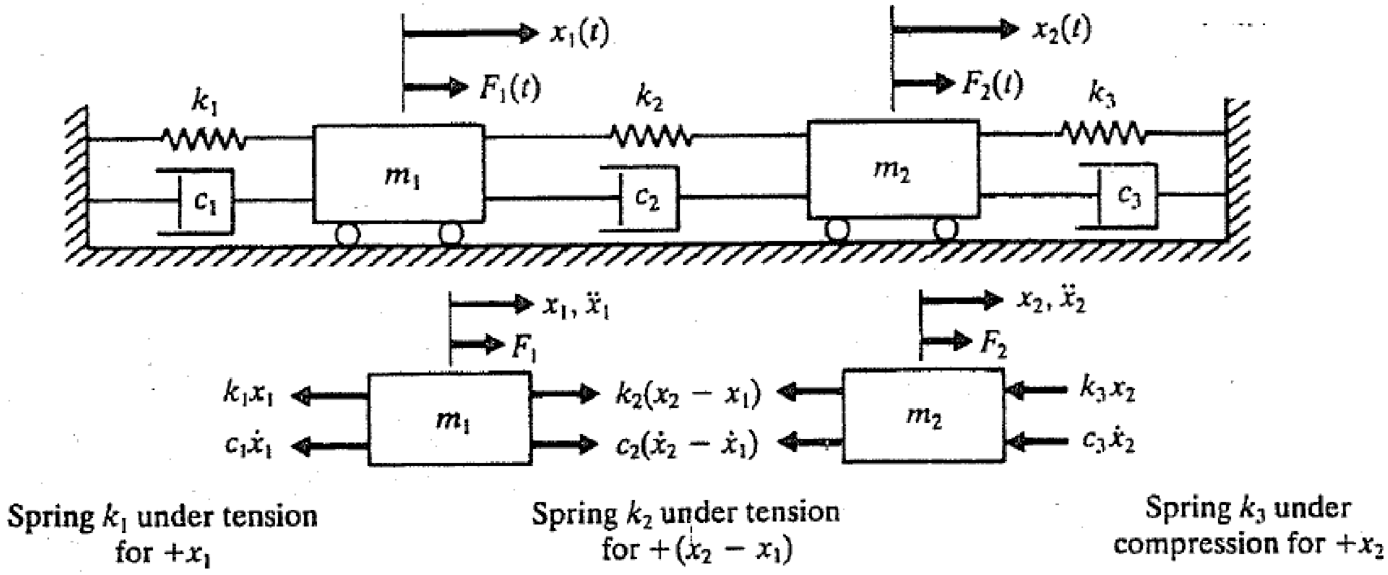


Equations of motion for forced vibration

- The external forces F_1 and F_2 act on the masses m_1 and m_2, respectively. The free body diagrams of the masses are shown in the figure.
- The application of Newton's second law of motion to each of the masses gives the equation of motion:

$$m_1\ddot{x}_1 + (c_1 + c_2)\,\dot{x}_1 - c_2\dot{x}_2 + (k_1 + k_2)\,x_1 - k_2x_2 = F_1$$

$$m_2\ddot{x}_2 - c_2\dot{x}_1 + (c_2 + c_3)\,\dot{x}_2 - k_2x_1 + (k_2 + k_3)\,x_2 = F_2$$

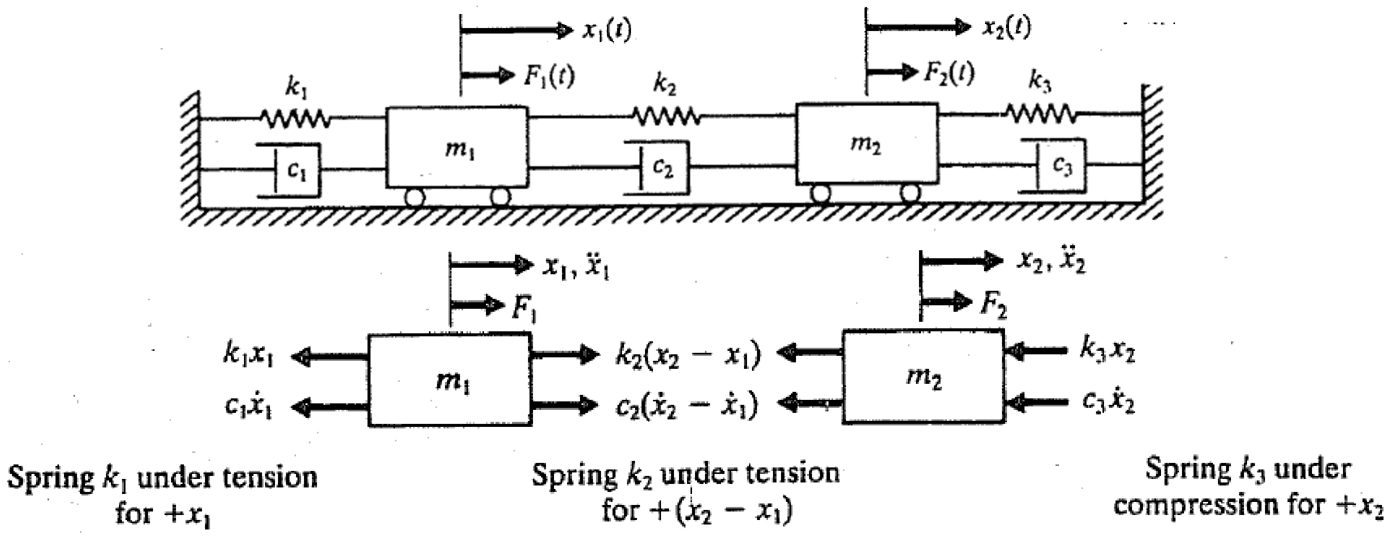


Equations of motion for forced vibration

- It can be seen that the first equation contains terms involving x_2, whereas the second equation contains terms involving x_1. Hence, they represent a system of two coupled second-order differential equations. We can therefore expect that the motion of the m_1 will influence the motion of m_2, and vica versa.

$$m_1\ddot{x}_1 + (c_1 + c_2)\,\dot{x}_1 - c_2\dot{x}_2 + (k_1 + k_2)\,x_1 - k_2x_2 = F_1$$

$$m_2\ddot{x}_2 - c_2\dot{x}_1 + (c_2 + c_3)\,\dot{x}_2 - k_2x_1 + (k_2 + k_3)\,x_2 = F_2$$



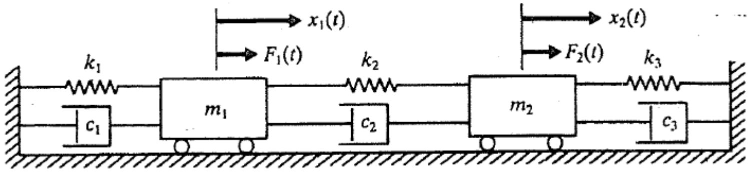
Equations of motion for forced vibration

- The equations can be written in matrix form as:

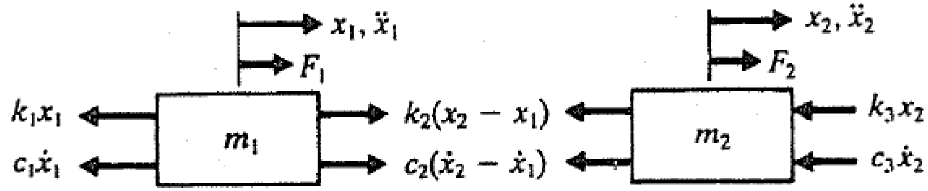
$$[m]\,\ddot{\vec{x}}(t) + [c]\,\dot{\vec{x}}(t) + [k]\,\vec{x}(t) = \vec{F}(t)$$

where [m], [c] and [k] are mass, damping and stiffness matrices, respectively and x(t) and F(t) are called the displacement and force vectors, respectively.which are given by:

$$[m] = \begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} \qquad [c] = \begin{bmatrix} c_1 + c_2 & -c_2 \\ -c_2 & c_2 + c_3 \end{bmatrix} \qquad [k] = \begin{bmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_2 + k_3 \end{bmatrix}$$



$$\vec{x}(t) = \begin{Bmatrix} x_1(t) \\ x_2(t) \end{Bmatrix} \qquad \vec{F}(t) = \begin{Bmatrix} F_1(t) \\ F_2(t) \end{Bmatrix}$$



Spring k_1 under tension for $+x_1$

Spring k_2 under tension for $+(x_2 - x_1)$

Spring k_3 under compression for $+x_2$

Equations of motion for forced vibration

- It can be seen that the matrices [m], [c] and [k] are all 2x2 matrices whose elements are the known masses, damping coefficienst, and stiffness of the system, respectively.
- Further, these matrices can be seen to be symmetric, so that:

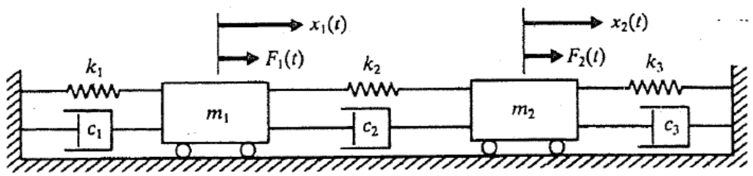
$$[m]^T = [m], \qquad [c]^T = [c], \qquad [k]^T = [k]$$

Free vibration analysis of an undamped system

- For the free vibration analysis of the system shown in the figure, we set $F_1(t)=F_2(t)=0$. Further, if the damping is disregarded, $c_1=c_2=c_3=0$, and the equations of motion reduce to:

$$m_1\ddot{x}_1(t) + (k_1 + k_2)x_1(t) - k_2x_2(t) = 0$$

$$m_2\ddot{x}_2(t) - k_2x_1(t) + (k_2 + k_3)x_2(t) = 0$$



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Free vibration analysis of an undamped system

- We are interested in knowing whether m_1 and m_2 can oscillate harmonically with the same frequency and phase angle but with different amplitudes. Assuming that it is possible to have harmonic motion of m_1 and m_2 at the same frequency ω and the same phase angle ϕ, we take the solutions to the equations

$$m_1\ddot{x}_1(t) + (k_1 + k_2)x_1(t) - k_2x_2(t) = 0$$

$$m_2\ddot{x}_2(t) - k_2x_1(t) + (k_2 + k_3)x_2(t) = 0$$

as:

$$x_1(t) = X_1 \cos(\omega t + \phi)$$

$$x_2(t) = X_2 \cos(\omega t + \phi)$$

where X_1 and X_2 are constants that denote the maximum amplitudes of $x_1(t)$ and $x_2(t)$ and ϕ is the phase angle.Substituting the above two solutions into the first two equations, we have:

Free vibration analysis of an undamped system

$$[\{-m_1\omega^2 + (k_1 + k_2)\} X_1 - k_2X_2]\cos(\omega t + \phi) = 0$$
$$[-k_2X_1 + \{-m_2\omega^2 + (k_2 + k_3)\} X_2]\cos(\omega t + \phi) = 0$$

- Since the above equations must be satisfied for all values of time t, the terms between brackets must be zero. This yields,

$$\{-m_1\omega^2 + (k_1 + k_2)\} X_1 - k_2X_2 = 0$$
$$-k_2X_1 + \{-m_2\omega^2 + (k_2 + k_3)\} X_2 = 0$$

which represents two simultaneous homogeneous algebraic equations in the unknowns X_1 and X_2. It can be seen that the above equation can be satisfied by the trivial soution $X_1=X_2=0$, which implies that there is no vibration. For a nontrivial solution of X_1 and X_2, the determinant of coefficients of X_1 and X_2 must be zero.

Free vibration analysis of an undamped system

$$\det \begin{bmatrix} \{-m_1\omega^2 + (k_1 + k_2)\} & -k_2 \\ -k_2 & \{m_2\omega^2 + (k_2 + k_3)\} \end{bmatrix} = 0$$

$$(m_1 m_2)\omega^4 - \{(k_1 + k_2)m_2 + (k_2 + k_3)m_1\}\omega^2 + \{(k_1 + k_2)(k_2 + k_3) - k_2^2\} = 0$$

- The above equation is called the **frequency** or **characteristic equation** because solution of this equation yields the frequencies of the characteristic values of the system. The roots of the above equation are given by:

$$\omega_1^2,\ \omega_2^2 = \frac{1}{2}\left\{\frac{(k_1 + k_2)m_2 + (k_2 + k_3)m_1}{m_1 m_2}\right\} \mp \frac{1}{2}\left[\left\{\frac{(k_1 + k_2)m_2 + (k_2 + k_3)m_1}{m_1 m_2}\right\}^2 - 4\left\{\frac{(k_1 + k_2)(k_2 + k_3) - k_2^2}{m_1 m_2}\right\}\right]^{1/2}$$

Free vibration analysis of an undamped system

- This shows that it is possible for the system to have a nontrivial harmonic solution of the form

$$x_1(t) = X_1 \cos(\omega t + \phi)$$

$$x_2(t) = X_2 \cos(\omega t + \phi)$$

when $\omega=\omega_1$ and $\omega=\omega_2$ given by:

$$\omega_1^2,\ \omega_2^2 = \frac{1}{2}\left\{\frac{(k_1 + k_2)m_2 + (k_2 + k_3)m_1}{m_1 m_2}\right\} \mp \frac{1}{2}\left[\left\{\frac{(k_1 + k_2)m_2 + (k_2 + k_3)m_1}{m_1 m_2}\right\}^2 - 4\left\{\frac{(k_1 + k_2)(k_2 + k_3) - k_2^2}{m_1 m_2}\right\}\right]^{1/2}$$

We shall denote the values of X_1 and X_2 corresponding to ω_1 as $X_1^{(1)}$ and $X_2^{(1)}$ and those corresponding to ω_2 as $X_1^{(2)}$ and $X_2^{(2)}$.

Free vibration analysis of an undamped system

- Further. since

$$\{-m_1\omega^2 + (k_1 + k_2)\} X_1 - k_2X_2 = 0$$

$$-k_2X_1 + \{-m_2\omega^2 + (k_2 + k_3)\} X_2 = 0$$

the above equation is homogeneous, only the ratios $r_1 = \{X_2^{(1)}/X_1^{(1)}\}$ and r2: $\{X_2^{(2)}/X_1^{(2)}\}$ can be found. For $\omega^2 = \omega_1^2$ and $\omega^2 = \omega_2^2$, the equations

$$\{-m_1\omega^2 + (k_1 + k_2)\} X_1 - k_2X_2 = 0$$

$$-k_2X_1 + \{-m_2\omega^2 + (k_2 + k_3)\} X_2 = 0$$

give:

$$r_1 = \frac{X_2^{(1)}}{X_1^{(1)}} = \frac{-m_1\omega_1^2 + (k_1 + k_2)}{k_2} = \frac{k_2}{-m_2\omega_1^2 + (k_2 + k_3)}$$

$$r_2 = \frac{X_2^{(2)}}{X_1^{(2)}} = \frac{-m_1\omega_2^2 + (k_1 + k_2)}{k_2} = \frac{k_2}{-m_2\omega_2^2 + (k_2 + k_3)}$$

- Notice that the two ratios are identical.

Free vibration analysis of an undamped system

- The normal modes of vibration corresponding to ω_1^2 and ω_2^2 can be expressed, respectively, as:

$$\vec{X}^{(1)} = \begin{Bmatrix} X_1^{(1)} \\ X_2^{(1)} \end{Bmatrix} = \begin{Bmatrix} X_1^{(1)} \\ r_1 X_1^{(1)} \end{Bmatrix} \qquad \vec{X}^{(2)} = \begin{Bmatrix} X_1^{(2)} \\ X_2^{(2)} \end{Bmatrix} = \begin{Bmatrix} X_1^{(2)} \\ r_2 X_1^{(2)} \end{Bmatrix}$$

- The vectors $\vec{X}^{(1)}$ and $\vec{X}^{(2)}$, which denote the normal modes of vibration are known as the **modal vectors of the system**. The free vibration solution or the motion in time can be expressed using

$$x_1(t) = X_1 \cos(\omega t + \phi)$$

$$x_2(t) = X_2 \cos(\omega t + \phi)$$

as:

$$\vec{x}^{(1)}(t) = \begin{Bmatrix} x_1^{(1)}(t) \\ x_2^{(1)}(t) \end{Bmatrix} = \begin{Bmatrix} X_1^{(1)} \cos(\omega_1 t + \phi_1) \\ r_1 X_1^{(1)} \cos(\omega_1 t + \phi_1) \end{Bmatrix} = \text{first mode}$$

$$\vec{x}^{(2)}(t) = \begin{Bmatrix} x_1^{(2)}(t) \\ x_2^{(2)}(t) \end{Bmatrix} = \begin{Bmatrix} X_1^{(2)} \cos(\omega_2 t + \phi_2) \\ r_2 X_1^{(2)} \cos(\omega_2 t + \phi_2) \end{Bmatrix} = \text{second mode}$$

where the constants $X_1^{(1)}$, $X_1^{(2)}$, ϕ_1, and ϕ_2 are determined by the initial conditions.

Free vibration analysis of an undamped system

Initial conditions:

Each of the two equations of motion ,

$$m_1\ddot{x}_1 + (c_1 + c_2)\,\dot{x}_1 - c_2\dot{x}_2 + (k_1 + k_2)\,x_1 - k_2x_2 = F_1$$

$$m_2\ddot{x}_2 - c_2\dot{x}_1 + (c_2 + c_3)\,\dot{x}_2 - k_2x_1 + (k_2 + k_3)\,x_2 = F_2$$

involves second order time derivatives; hence we need to specify two initial conditions for each mass.

The system can be made to vibrate in its ith normal mode (i=1,2) by subjecting it to the specific initial conditions.

$$x_1(t = 0) = X_1^{(i)} = \text{some constant}, \qquad \dot{x}_1(t = 0) = 0,$$

$$x_2(t = 0) = r_1X_1^{(i)}, \qquad \dot{x}_2(t = 0) = 0$$

However, for any other general initial conditions, both modes will be excited. The resulting motion, which is given by the general solution of the equations

$$m_1\ddot{x}_1(t) + (k_1 + k_2)x_1(t) - k_2x_2(t) = 0$$

$$m_2\ddot{x}_2(t) - k_2x_1(t) + (k_2 + k_3)x_2(t) = 0$$

can be obtained by a linear superposition of two normal modes.

Free vibration analysis of an undamped system

Initial conditions:

$$\vec{x}(t) = c_1\vec{x}_1(t) + c_2\vec{x}_2(t)$$

where c_1 and c_2 are constants.

Since $x_1^{(1)}(t)$ and $x_1^{(2)}(t)$ already involve the unknown constants $X_1^{(1)}$ and $X_1^{(2)}$ we can choose $c_1=c_2=1$ with no loss of generality. Thus, the components of the vector $\vec{x}(t)$ can be expressed as:

$$x_1(t) = x_1^{(1)}(t) + x_1^{(2)}(t) = X_1^{(1)} \cos(\omega_1 t + \phi_1) + X_1^{(2)} \cos(\omega_2 t + \phi_2)$$

$$x_2(t) = x_2^{(1)}(t) + x_2^{(2)}(t)$$

$$= r_1 X_1^{(1)} \cos(\omega_1 t + \phi_1) + r_2 X_1^{(2)} \cos(\omega_2 t + \phi_2)$$

where the unknown $X_1^{(1)}$, $X_1^{(2)}$, ϕ_1, and ϕ_2 can be determined from the initial conditions

$$x_1(t = 0) = x_1(0), \qquad \dot{x}_1(t = 0) = \dot{x}_1(0),$$

$$x_2(t = 0) = x_2(0), \qquad \dot{x}_2(t = 0) = \dot{x}_2(0)$$

Free vibration analysis of an undamped system

$$x_1(t) = x_1^{(1)}(t) + x_1^{(2)}(t) = X_1^{(1)} \cos(\omega_1 t + \phi_1) + X_1^{(2)} \cos(\omega_2 t + \phi_2)$$

$$\begin{aligned} x_2(t) &= x_2^{(1)}(t) + x_2^{(2)}(t) \\ &= r_1 X_1^{(1)} \cos(\omega_1 t + \phi_1) + r_2 X_1^{(2)} \cos(\omega_2 t + \phi_2) \end{aligned} \tag{5.15}$$

Thus if the initial conditions are given by

$$\begin{aligned} x_1(t = 0) &= x_1(0), \qquad \dot{x}_1(t = 0) = \dot{x}_1(0), \\ x_2(t = 0) &= x_2(0), \qquad \dot{x}_2(t = 0) = \dot{x}_2(0) \end{aligned} \tag{5.16}$$

the constants $X_1^{(1)}$, $X_1^{(2)}$, ϕ_1, and ϕ_2 can be found by solving the following equations (obtained by substituting Eqs. 5.16 into Eqs. 5.15):

$$\begin{aligned} x_1(0) &= X_1^{(1)} \cos \phi_1 + X_1^{(2)} \cos \phi_2 \\ \dot{x}_1(0) &= -\omega_1 X_1^{(1)} \sin \phi_1 - \omega_2 X_1^{(2)} \sin \phi_2 \\ x_2(0) &= r_1 X_1^{(1)} \cos \phi_1 + r_2 X_1^{(2)} \cos \phi_2 \\ \dot{x}_2(0) &= -\omega_1 r_1 X_1^{(1)} \sin \phi_1 - \omega_2 r_2 X_1^{(2)} \sin \phi_2 \end{aligned} \tag{5.17}$$

Free vibration analysis of an undamped system

Equations (5.17) can be regarded as four algebraic equations in the unknowns $X_1^{(1)} \cos \phi_1$, $X_1^{(2)} \cos \phi_2$, $X_1^{(1)} \sin \phi_1$, and $X_1^{(2)} \sin \phi_2$. The solution of Eqs. (5.17) can be expressed as

$$X_1^{(1)} \cos \phi_1 = \left\{ \frac{r_2 x_1(0) - x_2(0)}{r_2 - r_1} \right\}, \qquad X_1^{(2)} \cos \phi_2 = \left\{ \frac{-r_1 x_1(0) + x_2(0)}{r_2 - r_1} \right\}$$

$$X_1^{(1)} \sin \phi_1 = \left\{ \frac{-r_2 \dot{x}_1(0) + \dot{x}_2(0)}{\omega_1 (r_2 - r_1)} \right\}, \qquad X_1^{(2)} \sin \phi_2 = \left\{ \frac{r_1 \dot{x}_1(0) - \dot{x}_2(0)}{\omega_2 (r_2 - r_1)} \right\}$$

from which we obtain the desired solution

$$X_1^{(1)} = [\{X_1^{(1)} \cos \phi_1\}^2 + \{X_1^{(1)} \sin \phi_1\}^2]^{1/2}$$

$$= \frac{1}{(r_2 - r_1)} \left[\{r_2 x_1(0) - x_2(0)\}^2 + \frac{\{-r_2 \dot{x}_1(0) + \dot{x}_2(0)\}^2}{\omega_1^2} \right]^{1/2}$$

$$X_1^{(2)} = [\{X_1^{(2)} \cos \phi_2\}^2 + \{X_1^{(2)} \sin \phi_2\}^2]^{1/2}$$

$$= \frac{1}{(r_2 - r_1)} \left[\{-r_1 x_1(0) + x_2(0)\}^2 + \frac{\{r_1 \dot{x}_1(0) - \dot{x}_2(0)\}^2}{\omega_2^2} \right]^{1/2}$$

Free vibration analysis of an undamped system

from which we obtain the desired solution

$$X_1^{(1)} = [\{X_1^{(1)} \cos \phi_1\}^2 + \{X_1^{(1)} \sin \phi_1\}^2]^{1/2}$$

$$= \frac{1}{(r_2 - r_1)}\left[\{r_2 x_1(0) - x_2(0)\}^2 + \frac{\{-r_2 \dot{x}_1(0) + \dot{x}_2(0)\}^2}{\omega_1^2} \right]^{1/2}$$

$$X_1^{(2)} = [\{X_1^{(2)} \cos \phi_2\}^2 + \{X_1^{(2)} \sin \phi_2\}^2]^{1/2}$$

$$= \frac{1}{(r_2 - r_1)}\left[\{-r_1 x_1(0) + x_2(0)\}^2 + \frac{\{r_1 \dot{x}_1(0) - \dot{x}_2(0)\}^2}{\omega_2^2} \right]^{1/2}$$

$$\phi_1 = \tan^{-1}\left\{\frac{X_1^{(1)} \sin \phi_1}{X_1^{(1)} \cos \phi_1}\right\} = \tan^{-1}\left\{\frac{-r_2 \dot{x}_1(0) + \dot{x}_2(0)}{\omega_1[r_2 x_1(0) - x_2(0)]}\right\}$$

$$\phi_2 = \tan^{-1}\left\{\frac{X_1^{(2)} \sin \phi_2}{X_1^{(2)} \cos \phi_2}\right\} = \tan^{-1}\left\{\frac{r_1 \dot{x}_1(0) - \dot{x}_2(0)}{\omega_2[-r_1 x_1(0) + x_2(0)]}\right\}$$

Frequencies of a mass-spring system

Example: Find the natural frequencies and mode shapes of a spring mass system , which is constrained to move in the vertical direction.

Solution: The equations of motion are given by:

$$m\ddot{x}_1 + 2kx_1 - kx_2 = 0$$

$$m\ddot{x}_2 - kx_1 + 2kx_2 = 0$$

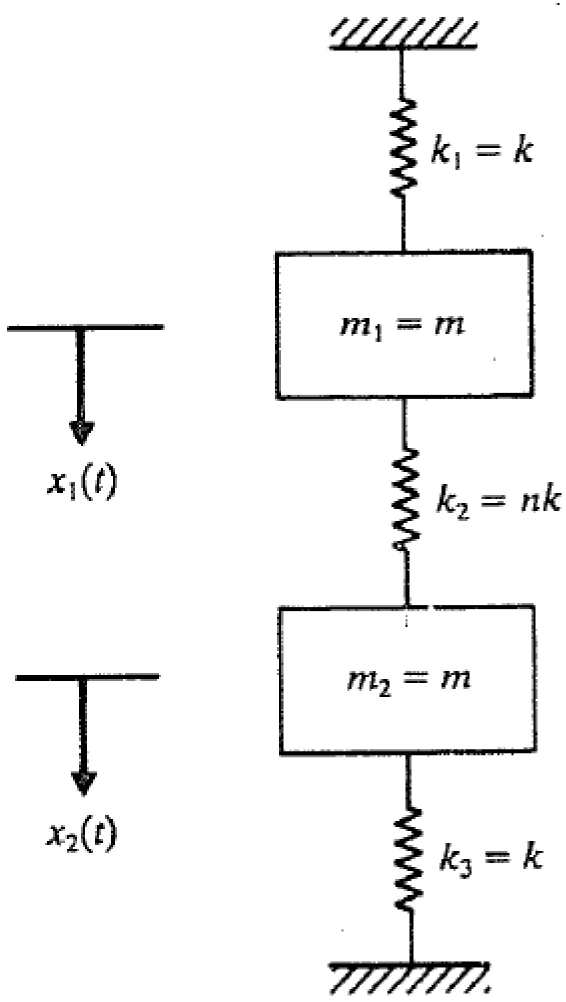
By assuming harmonic solution as:

$$x_i(t) = X_i \cos(\omega t + \phi);\ i = 1, 2$$

the frequency equation can be obtained by:

$$\begin{vmatrix} (-m\omega^2 + 2k) & (-k) \\ (-k) & (-m\omega^2 + 2k) \end{vmatrix} = 0$$

$$m^2\omega^4 - 4km\omega^2 + 3k^2 = 0$$



Frequencies of a mass-spring system

$$\begin{vmatrix} (-m\omega^2 + 2k) & (-k) \\ (-k) & (-m\omega^2 + 2k) \end{vmatrix} = 0$$

$$m^2\omega^4 - 4km\omega^2 + 3k^2 = 0$$

- The solution to the above equation gives the natural frequencies:

$$\omega_1 = \left\{ \frac{4km - [16k^2m^2 - 12m^2k^2]^{1/2}}{2m^2} \right\}^{1/2} = \sqrt{\frac{k}{m}}$$

$$\omega_2 = \left\{ \frac{4km + [16k^2m^2 - 12m^2k^2]^{1/2}}{2m^2} \right\}^{1/2} = \sqrt{\frac{3k}{m}}$$

Frequencies of a mass-spring system

- From

$$r_1 = \frac{X_2^{(1)}}{X_1^{(1)}} = \frac{-m_1\omega_1^2 + (k_1 + k_2)}{k_2} = \frac{k_2}{-m_2\omega_1^2 + (k_2 + k_3)}$$

$$r_2 = \frac{X_2^{(2)}}{X_1^{(2)}} = \frac{-m_1\omega_2^2 + (k_1 + k_2)}{k_2} = \frac{k_2}{-m_2\omega_2^2 + (k_2 + k_3)}$$

the amplitude ratios are given by:

$$r_1 = \frac{X_2^{(1)}}{X_1^{(1)}} = \frac{-m\omega_1^2 + 2k}{k} = \frac{k}{-m\omega_1^2 + 2k} = 1$$

$$r_2 = \frac{X_2^{(2)}}{X_1^{(2)}} = \frac{-m\omega_2^2 + 2k}{k} = \frac{k}{-m\omega_2^2 + 2k} = -1$$

Frequencies of a mass-spring system

- From $$\vec{x}^{(1)}(t) = \begin{Bmatrix} x_1^{(1)}(t) \\ x_2^{(1)}(t) \end{Bmatrix} = \begin{Bmatrix} X_1^{(1)} \cos(\omega_1 t + \phi_1) \\ r_1 X_1^{(1)} \cos(\omega_1 t + \phi_1) \end{Bmatrix} = \text{first mode}$$
$$\vec{x}^{(2)}(t) = \begin{Bmatrix} x_1^{(2)}(t) \\ x_2^{(2)}(t) \end{Bmatrix} = \begin{Bmatrix} X_1^{(2)} \cos(\omega_2 t + \phi_2) \\ r_2 X_1^{(2)} \cos(\omega_2 t + \phi_2) \end{Bmatrix} = \text{second mode}$$

- The natural modes are given by

$$\text{First mode} = \vec{x}^{(1)}(t) = \begin{Bmatrix} X_1^{(1)} \cos\left(\sqrt{\dfrac{k}{m}}\, t + \phi_1\right) \\ X_1^{(1)} \cos\left(\sqrt{\dfrac{k}{m}}\, t + \phi_1\right) \end{Bmatrix}$$

$$\text{Second mode} = \vec{x}^{(2)}(t) = \begin{Bmatrix} X_1^{(2)} \cos\left(\sqrt{\dfrac{3k}{m}}\, t + \phi_2\right) \\ -X_1^{(2)} \cos\left(\sqrt{\dfrac{3k}{m}}\, t + \phi_2\right) \end{Bmatrix}$$

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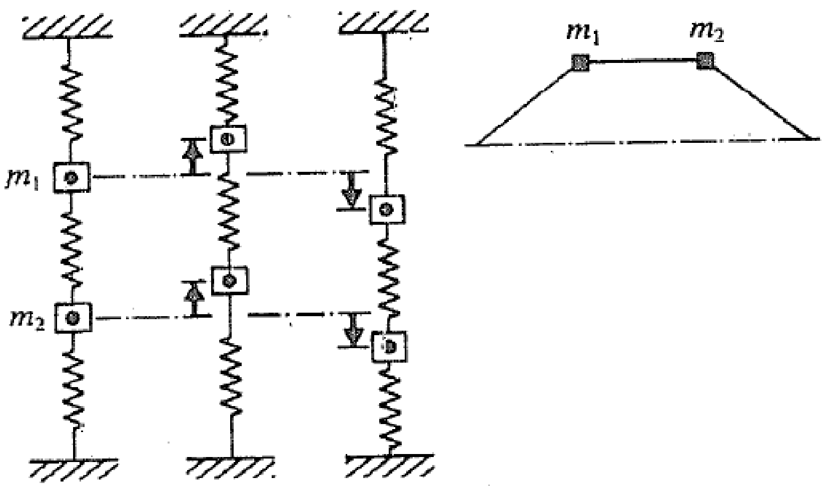
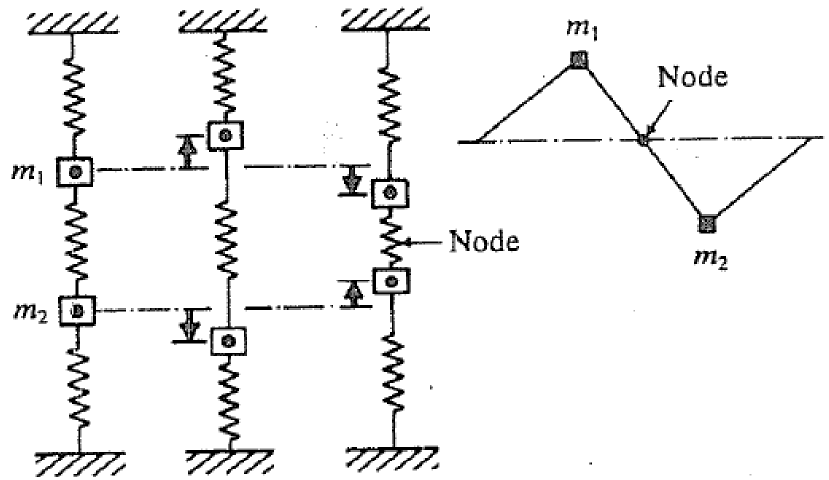


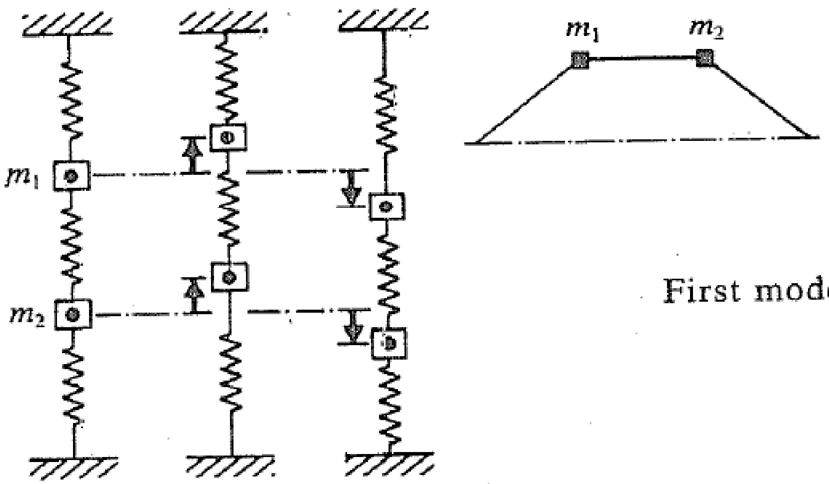
(a) First mode



(b) Second mode

Frequencies of a mass-spring system

- It can be seen that when the system vibrates in its first mode, the amplitudes of the two masses remain the same. This implies that the length of the middle spring remains constant. Thus the motions of the mass 1 and mass 2 are in phase.



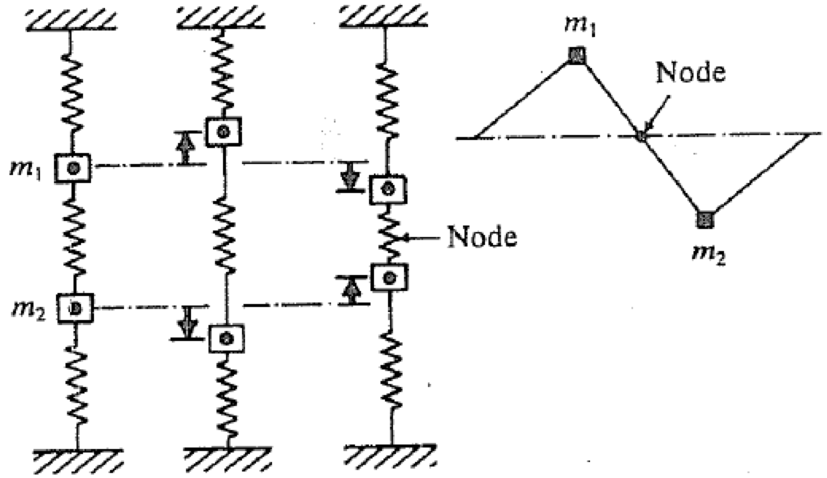
(a) First mode

$$\text{First mode} = \vec{x}^{(1)}(t) = \begin{Bmatrix} X_1^{(1)} \cos\left(\sqrt{\dfrac{k}{m}}\, t + \phi_1\right) \\ X_1^{(1)} \cos\left(\sqrt{\dfrac{k}{m}}\, t + \phi_1\right) \end{Bmatrix}$$

$$\text{Second mode} = \vec{x}^{(2)}(t) = \begin{Bmatrix} X_1^{(2)} \cos\left(\sqrt{\dfrac{3k}{m}}\, t + \phi_2\right) \\ -X_1^{(2)} \cos\left(\sqrt{\dfrac{3k}{m}}\, t + \phi_2\right) \end{Bmatrix}$$

Frequencies of a mass-spring system

- When the system vibrates in its second mode, the equations below show that the displacements of the two masses have the same magnitude with opposite signs. Thus the motions of the mass 1 and mass 2 are out of phase. In this case, the midpoint of the middle spring remains stationary for all time. Such a point is called a **node**.



(b) Second mode

$$\vec{x}^{(1)}(t) = \begin{Bmatrix} X_1^{(1)} \cos\left(\sqrt{\frac{k}{m}}\,t + \phi_1\right) \\ X_1^{(1)} \cos\left(\sqrt{\frac{k}{m}}\,t + \phi_1\right) \end{Bmatrix}$$

$$\vec{x}^{(2)}(t) = \begin{Bmatrix} X_1^{(2)} \cos\left(\sqrt{\frac{3k}{m}}\,t + \phi_2\right) \\ -X_1^{(2)} \cos\left(\sqrt{\frac{3k}{m}}\,t + \phi_2\right) \end{Bmatrix}$$

Frequencies of a mass-spring system

- Using equations

$$x_1(t) = x_1^{(1)}(t) + x_1^{(2)}(t) = X_1^{(1)} \cos(\omega_1 t + \phi_1) + X_1^{(2)} \cos(\omega_2 t + \phi_2)$$

$$x_2(t) = x_2^{(1)}(t) + x_2^{(2)}(t) = r_1 X_1^{(1)} \cos(\omega_1 t + \phi_1) + r_2 X_1^{(2)} \cos(\omega_2 t + \phi_2)$$

$$\text{First mode} = \vec{x}^{(1)}(t) = \begin{Bmatrix} X_1^{(1)} \cos\left(\sqrt{\dfrac{k}{m}}\, t + \phi_1\right) \\ X_1^{(1)} \cos\left(\sqrt{\dfrac{k}{m}}\, t + \phi_1\right) \end{Bmatrix}$$

$$\text{Second mode} = \vec{x}^{(2)}(t) = \begin{Bmatrix} X_1^{(2)} \cos\left(\sqrt{\dfrac{3k}{m}}\, t + \phi_2\right) \\ -X_1^{(2)} \cos\left(\sqrt{\dfrac{3k}{m}}\, t + \phi_2\right) \end{Bmatrix}$$

the motion (general solution) of the system can be expressed as:

$$x_1(t) = X_1^{(1)} \cos\left(\sqrt{\frac{k}{m}}\, t + \phi_1\right) + X_1^{(2)} \cos\left(\sqrt{\frac{3k}{m}}\, t + \phi_2\right)$$

$$x_2(t) = X_1^{(1)} \cos\left(\sqrt{\frac{k}{m}}\, t + \phi_1\right) - X_1^{(2)} \cos\left(\sqrt{\frac{3k}{m}}\, t + \phi_2\right)$$

Forced vibration analysis

- The equation of motion of a general two degree of freedom system under external forces can be written as:

$$\begin{bmatrix} m_{11} & m_{12} \\ m_{12} & m_{22} \end{bmatrix} \begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{Bmatrix} + \begin{bmatrix} c_{11} & c_{12} \\ c_{12} & c_{22} \end{bmatrix} \begin{Bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{Bmatrix} + \begin{bmatrix} k_{11} & k_{12} \\ k_{12} & k_{22} \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} F_1 \\ F_2 \end{Bmatrix}$$

- We shall consider the external forces to be harmonic:

$$F_j(t) = F_{j0}e^{i\omega t}, \qquad j = 1, 2$$

where ω is the forcing frequency. We can write the steady state solution as:

$$x_j(t) = X_j e^{i\omega t}, \qquad j = 1, 2$$

where X_1 and X_2 are, in general, complex quantities that depend on ω and the system parameters. Substituting the above two equations into the first one:

Forced vibration analysis

- We obtain: $$\begin{bmatrix} (-\omega^2 m_{11} + i\omega c_{11} + k_{11}) & (-\omega^2 m_{12} + i\omega c_{12} + k_{12}) \\ (-\omega^2 m_{12} + i\omega c_{12} + k_{12}) & (-\omega_2 m_{22} + i\omega c_{22} + k_{22}) \end{bmatrix} \begin{Bmatrix} X_1 \\ X_2 \end{Bmatrix}$$
$$= \begin{Bmatrix} F_{10} \\ F_{20} \end{Bmatrix}$$
(

- If we define a term called 'mechanical impedance' $Z_{rs}(i\omega)$ as:

$$Z_{rs}(i\omega) = -\omega^2 m_{rs} + i\omega c_{rs} + k_{rs}, \qquad r, s = 1, 2$$

and write the first equation as: $[Z(i\omega)]\vec{X} = \vec{F}_0$

where

$$[Z(i\omega)] = \begin{bmatrix} Z_{11}(i\omega) & Z_{12}(i\omega) \\ Z_{12}(i\omega) & Z_{22}(i\omega) \end{bmatrix} = \text{Impedance matrix}$$

$$\vec{X} = \begin{Bmatrix} X_1 \\ X_2 \end{Bmatrix} \qquad \vec{F}_0 = \begin{Bmatrix} F_{10} \\ F_{20} \end{Bmatrix}$$

Forced vibration analysis

- The equation $[Z(i\omega)]\vec{X} = \vec{F}_0$

  can be solved to obtain: $X = [Z(i\omega)]^{-1}\vec{F}_0$
- Where the inverse of the impedance matrix is given by:

$$[Z(i\omega)]^{-1} = \frac{1}{Z_{11}(i\omega)Z_{22}(i\omega) - Z_{12}^2(i\omega)}\begin{bmatrix} Z_{22}(i\omega) & -Z_{12}(i\omega) \\ -Z_{12}(i\omega) & Z_{11}(i\omega) \end{bmatrix}$$

- Therefore, the solutions are:

$$X_1(i\omega) = \frac{Z_{22}(i\omega)F_{10} - Z_{12}(i\omega)F_{20}}{Z_{11}(i\omega)Z_{22}(i\omega) - Z_{12}^2(i\omega)}$$

$$X_2(i\omega) = \frac{-Z_{12}(i\omega)F_{10} + Z_{11}(i\omega)F_{20}}{Z_{11}(i\omega)Z_{22}(i\omega) - Z_{12}^2(i\omega)}$$

- By substituting these into the below equation, the solutions can be obtained.

$$x_j(t) = X_j e^{i\omega t}, \qquad j = 1, 2$$

Multi-degree of freedom systems

- Modeling of continuous systems as multidegree of freedom systems
- Eigenvalue problem

Multidegree of freedom systems

- As stated before, most engineering systems are **continuous** and have an **infinite number of degrees of freedom**. The vibration analysis of continuous systems requires the solution of partial differential equations, which is quite difficult.

- In fact, analytical solutions do not exist for many partial differential equations. The analysis of a multidegree of freedom system on the other hand, requires the solution of a set of ordinary differential equations, which is relatively simple. Hence, for simplicity of analysis, **continuous systems** are often approximated as **multidegree of freedom systems**.

- For a system having n degrees of freedom, there are n associated natural frequencies, each associated with its own mode shape.

Multidegree of freedom systems

- Different methods can be used to **approximate a continuous system** as a **multidegree of freedom system**. A simple method involves **replacing the distributed mass** or inertia of the system **by a finite number of lumped masses** or rigid bodies.

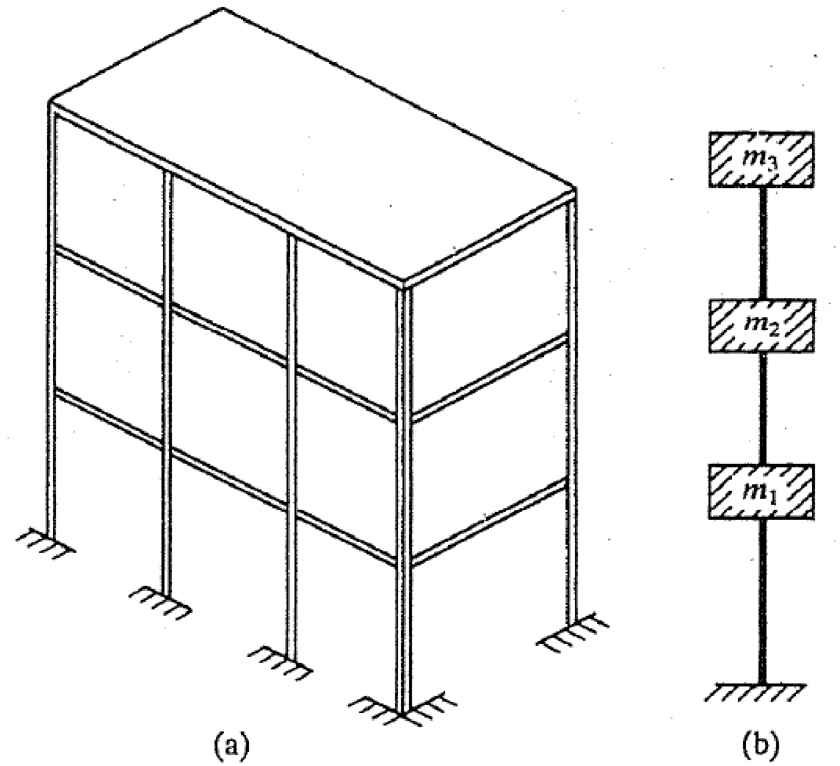
- The lumped masses are assumed to be connected by massless elastic and damping members.

- Linear coordinates are used to describe the motion of the lumped masses. Such models are called **lumped parameter of lumped mass or discrete mass systems**.

- The **minimum number of coordinates** necessary to describe the motion of the lumped masses and rigid bodies defines **the number of degrees of freedom** of the system. Naturally, the larger the number of lumped masses used in the model, the higher the accuracy of the resulting analysis.

Multidegree of freedom systems

- Some problems automatically indicate the type of lumped parameter model to be used.
- For example, the three storey building shown in the figure automatically suggests using a three lumped mass model as indicated in the figure.
- In this model, the inertia of the system is assumed to be concentrated as three point masses located at the floor levels, and the elasticities of the columns are replaced by the springs.



Multidegree of freedom systems

- Another popular method of approximating a continuous system as a multidegree of freedom system involves replacing the geometry of the system by a large number of small elements.
- By assuming a simple solution within each element, the principles of **compatibility** and **equilibrium** are used to find an approximate solution to the original system. This method is known as the **finite element method**.

Using Newton's second law to derive equations of motion

The following procedure can be adopted to derive the equations of motion of a multidegree of freedom system using Newton's second law of motion.

1. Set up suitable coordinates to describe the positions of the various point masses and rigid bodies in the system. Assume suitable positive directions for the displacements, velocities and accelerations of the masses and rigid bodies.

2. Determine the static equilibrium configuration of the system and measure the displacements of the masses and rigid bodies from their respective static equilibrium positions.

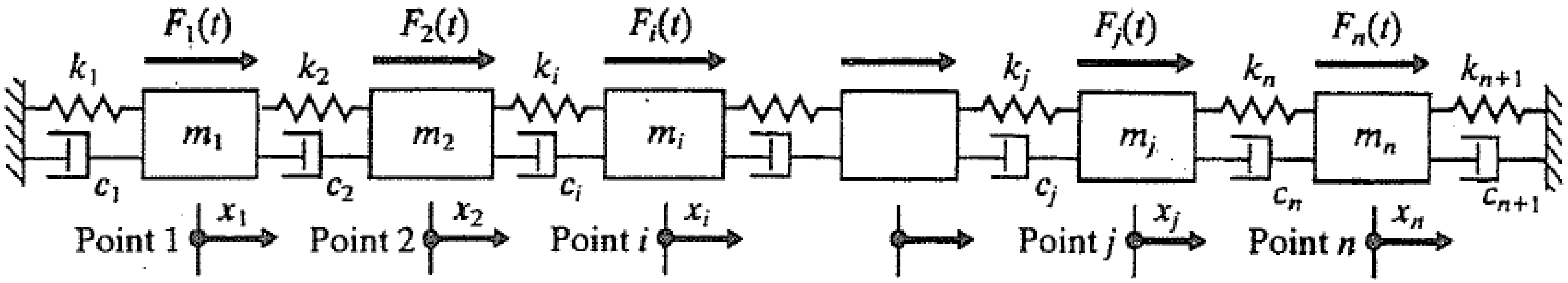
3. Draw the free body diagram of each mass or rigid body in the system. Indicate the spring, damping and external forces acting on each mass or rigid body when positive displacement or velocity are given to that mass or rigid body.

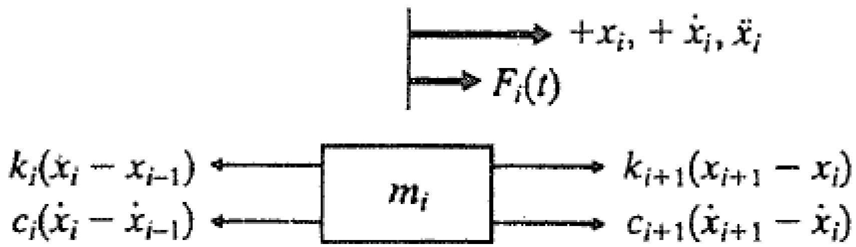
Using Newton's second law to derive equations of motion

4. Apply Newton's second law of motion to each mass or rigid body shown by the free body diagram as:

$$m_i \ddot{x}_i = \sum_j F_{ij} \text{ (for mass } m_i\text{)}$$

Example: Derive the equations of motion of the spring-mass-damper system shown in the figure.





Using Newton's second law to derive equations of motion

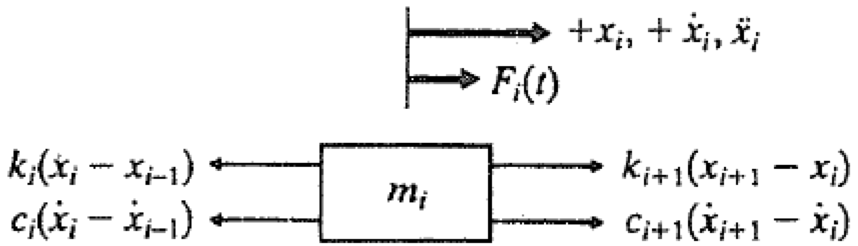
- Draw free-body diagrams of masses and apply Newton's second law of motion. The coordinates describing the positions of the masses, $x_i(t)$, are measured from their respective static equilibrium positions, as indicated in the figure. The application of the Newton's second law of motion to mass m_i gives:

$$m_i\ddot{x}_i = -k_i(x_i - x_{i-1}) + k_{i+1}(x_{i+1} - x_i) - c_i(\dot{x}_i - \dot{x}_{i-1}) + c_{i+1}(\dot{x}_{i+1} - \dot{x}_i) + F_i\,;\; i = 2, 3, \ldots, n-1$$

- or

$$m_i\ddot{x}_i - c_i\dot{x}_{i-1} + (c_i + c_{i+1})\dot{x}_i - c_{i+1}\dot{x}_{i+1} - k_i x_{i-1} + (k_i + k_{i+1})x_i - k_{i+1}x_{i+1} = F_i\,;\; i = 2, 3, \ldots, n-1$$

- The equations of motion of the masses m_1 and m_2 can be derived from the above equations by setting i=1 along with x_o=0 and i=n along with x_{n+1}=0, respectively.



Equations of motion in matrix form

- The equations of motion in matrix form in the above example can be expressed as:

$$[m]\,\ddot{\vec{x}} + [c]\,\dot{\vec{x}} + [k]\,\vec{x} = \vec{F}$$

where $[m]$, $[c]$, and $[k]$ are called the mass, damping, and stiffness matrices, respectively, and are given by

$$[m] = \begin{bmatrix} m_1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & m_2 & 0 & \cdots & 0 & 0 \\ 0 & 0 & m_3 & \cdots & 0 & 0 \\ \cdot & & & & & \\ \cdot & & & & & \\ \cdot & & & & & \\ 0 & 0 & 0 & \cdots & 0 & m_n \end{bmatrix}$$

$$[c] = \begin{bmatrix} (c_1 + c_2) & -c_2 & 0 & \cdots & 0 & 0 \\ -c_2 & (c_2 + c_3) & -c_3 & \cdots & 0 & 0 \\ 0 & -c_3 & (c_3 + c_4) & \cdots & 0 & 0 \\ \cdot & \cdot & \cdot & \cdots & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdots & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdots & \cdot & \cdot \\ 0 & 0 & 0 & \cdots & -c_n & (c_n + c_{n+1}) \end{bmatrix}$$

Equations of motion in matrix form

$$[k] = \begin{bmatrix} (k_1 + k_2) & -k_2 & 0 & \cdots & 0 & 0 \\ -k_2 & (k_2 + k_3) & -k_3 & \cdots & 0 & 0 \\ 0 & -k_3 & (k_3 + k_4) & \cdots & 0 & 0 \\ \cdot & & & & & \\ \cdot & & & & & \\ \cdot & & & & & \\ 0 & 0 & 0 & \cdots & -k_n & (k_n + k_{n+1}) \end{bmatrix}$$

and $\vec{x}$, $\dot{\vec{x}}$, $\ddot{\vec{x}}$, and $\vec{F}$ are the displacement, velocity, acceleration, and force vectors, given by

$$\vec{x} = \begin{Bmatrix} x_1(t) \\ x_2(t) \\ \cdot \\ \cdot \\ \cdot \\ x_n(t) \end{Bmatrix}, \qquad \dot{\vec{x}} = \begin{Bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \\ \cdot \\ \cdot \\ \cdot \\ \dot{x}_n(t) \end{Bmatrix},$$

$$\ddot{\vec{x}} = \begin{Bmatrix} \ddot{x}_1(t) \\ \ddot{x}_2(t) \\ \cdot \\ \cdot \\ \cdot \\ \ddot{x}_n(t) \end{Bmatrix}, \qquad \vec{F} = \begin{Bmatrix} F_1(t) \\ F_2(t) \\ \cdot \\ \cdot \\ \cdot \\ F_n(t) \end{Bmatrix}$$

Equations of motion in matrix form

- For an undamped system, the equations of motion reduce to:

$$[m]\,\ddot{\vec{x}} + [k]\,\vec{x} = \vec{F}$$

- The differential equations of the spring-mass system considered in the example, can be seen to be coupled. Each equation involves more than one coordinate. This means that the equations can not be solved individually one at a time; they can only be solved simultaneously.

- In addition, the system can be seen to be statically coupled since stiffnesses are coupled- that is the stiffness matrix has at least one nonzero off-diagonal term. On the other hand, if the mass matrix has at least one off-diagonal term nonzero, the system is said to be dynamically coupled. Further, if both the stiffness and the mass matrices have nonzero off-diagonal terms, the system is said to be coupled both statically and dynamically.

Undamped free vibrations

- The equations of motion for a freely vibrating undamped system can be obtained by omitting the damping matrix and applied load vector from:

$$\mathbf{m}\ddot{\mathbf{x}} + \mathbf{c}\dot{\mathbf{x}} + \mathbf{k}\mathbf{x} = \mathbf{0}$$

  in which **0** is a zero vector. The problem of vibration analysis consists of determining the conditions under which the equilibrium condition expressed by the above equation will be satisfied.

- By analogy with the behaviour of SDOF systems, it will be assumed that the free-vibration motion is simple harmonic (the first equation below), which may be expressed for a multi degree of freedom system as:

$$\mathbf{x}(t) = \hat{\mathbf{x}}\sin(\omega t + \theta)$$

$$\ddot{\mathbf{x}} = -\omega^2 \hat{\mathbf{x}}\sin(\omega t + \theta) = -\omega^2 \mathbf{x}$$

- In the above expressions, $\hat{\mathbf{x}}$ represents the shape of the system (which does not change with time; only the amplitude varies) and θ is a phase angle. The third equation above represents the accelerations in the free vibration.

Undamped free vibrations

- Substituting

$$\mathbf{x}(t) = \hat{\mathbf{x}} \sin(\omega t + \theta)$$
$$\ddot{\mathbf{x}} = -\omega^2 \hat{\mathbf{x}} \sin(\omega t + \theta) = -\omega^2 \mathbf{x}$$

in the equation $\mathbf{m}\ddot{\mathbf{x}} + \mathbf{c}\dot{\mathbf{x}} + \mathbf{k}\mathbf{x} = \mathbf{0}$

we obtain: $-\omega^2 \mathbf{m}\hat{\mathbf{x}} \sin(\omega t + \theta) + \mathbf{k}\hat{\mathbf{x}} \sin(\omega t + \theta) = \mathbf{0}$

which (since the sine term is arbitrary and may be omitted) may be written:

$$\left[\mathbf{k} - \omega^2 \mathbf{m}\right]\hat{\mathbf{x}} = \mathbf{0}$$

- The above equation is one way of expressing what is called an **eigenvalue or characteristic value problem**. The quantities ω^2 are the **eigenvalues or characteristic values indicating the square of the free-vibration frequencies**, while the corresponding displacement vectors $\hat{\mathbf{x}}$ express the corresponding shapes of the vibrating system- known as the **eigenvectors or mode shapes**.

Undamped free vibrations

- It can be shown by Cramer's rule that the solution of this set of simultaneous equations is of the form:

$$\hat{\mathbf{x}} = \frac{\mathbf{0}}{\left\| \mathbf{k} - \omega^2 \mathbf{m} \right\|}$$

- Hence a nontrivial solution is possible only when the denominator determinant vanishes. In other words, finite amplitude free vibrations are possible only when

$$\left\| \mathbf{k} - \omega^2 \mathbf{m} \right\| = 0$$

- The above equation is called the frequency equation of the system. Expanding the determinant will give an algebraic equation of the Nth degree in the frequency parameter ω^2 for a system having N degrees of freedom.
- The N roots of this equation $\left(\omega_1^2, \omega_2^2, \omega_3^2, \ldots., \omega_N^2\right)$ represent the frequencies of the N modes of vibration which are possible in the system.

Undamped free vibrations

- The mode having the lowest frequency is called the first mode, the next higher frequency is the second mode, etc.
- The vector made up of the entire set of modal frequencies, arranged in sequence, will be called the frequency vector ω.

$$\omega = \begin{Bmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \\ \vdots \\ \omega_N \end{Bmatrix}$$

Normalization:

It was noted earlier that the vibration mode amplitudes obtained from the eigenproblem solution are arbitrary; any amplitude will satisfy the basic frequency equation

$$\left\| \mathbf{k} - \omega^2 \mathbf{m} \right\| = 0$$

and only the resulting shapes are uniquely defined.

Normalization of modes

- In the analysis process described above, the amplitude of one degree of freedom (the first actually) has been set to unity, and the other displacements have been determined relative to this reference value. This is called **normalizing the mode shapes** with respect to the specified reference coordinate.

- Other normalizing procedures also are frequently used; e.g., in many computer programs, the shapes are normalized relative to the maximum displacement value in each mode rather than with respect to any particular coordinate. Thus, the maximum value in each modal vector is unity, which provides convenient numbers for use in subsequent calculations.

Normalization of modes

- The normalizing procedure most often used in computer programs for structural vibration analysis, however, involves adjusting each modal amplitude to the amplitude $\hat{\phi}_n$ which satisfies the condition

$$\hat{\phi}_n^T \mathbf{m} \hat{\phi}_n = 1$$

- This can be accomplished by computing the scalar factor

$$\hat{\mathbf{v}}_\mathbf{n}^\mathbf{T} \mathbf{m} \hat{\mathbf{v}}_\mathbf{m} = \hat{M}_n$$

where $\hat{\mathbf{v}}_\mathbf{n}$ represents an arbitrarily determined modal amplitude, and then computing the normalized mode shapes as follows:

$$\hat{\phi}_n = \hat{\mathbf{v}}_\mathbf{n} \hat{M}_n^{-1/2}$$

By simple substitution, it is easy to show that this givs the desired result. A consequence of this type of normalizing together with the modal orthogonality relationships relative to the mass matrix is that

$$\hat{\boldsymbol{\varphi}}_\mathbf{n}^\mathbf{T} \mathbf{m} \hat{\boldsymbol{\varphi}}_\mathbf{n} = \mathbf{I}$$

where ϕ is the complete set of N normalized mode shapes and I is an NxN identity matrix. The mode shapes normalized in this fashion are said to be orthonormal relative to the mass matrix.

Mode shapes of a four storey 2D frame

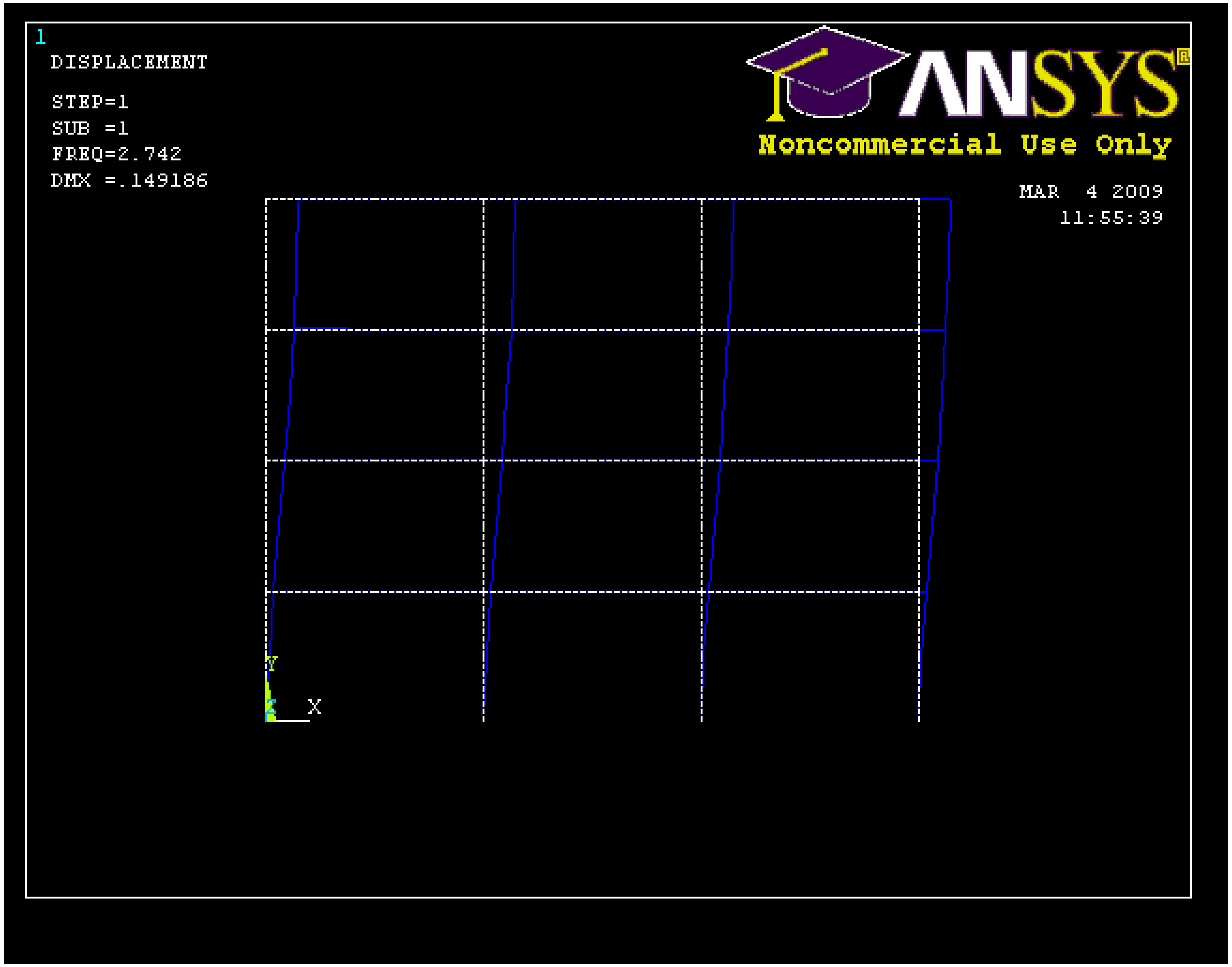
- A model of a four-story three-bay frame can be evaluated to determine the mode shapes. This 2 D model is from a typical building from the Marmara region in Turkey.

- Generally, the first mode of vibration is the one of primary interest. The first mode usually has the largest contribution to the structure's motion. The period of this mode is the longest and the natural frequency is the lowest.

- Please click on the movie to start!

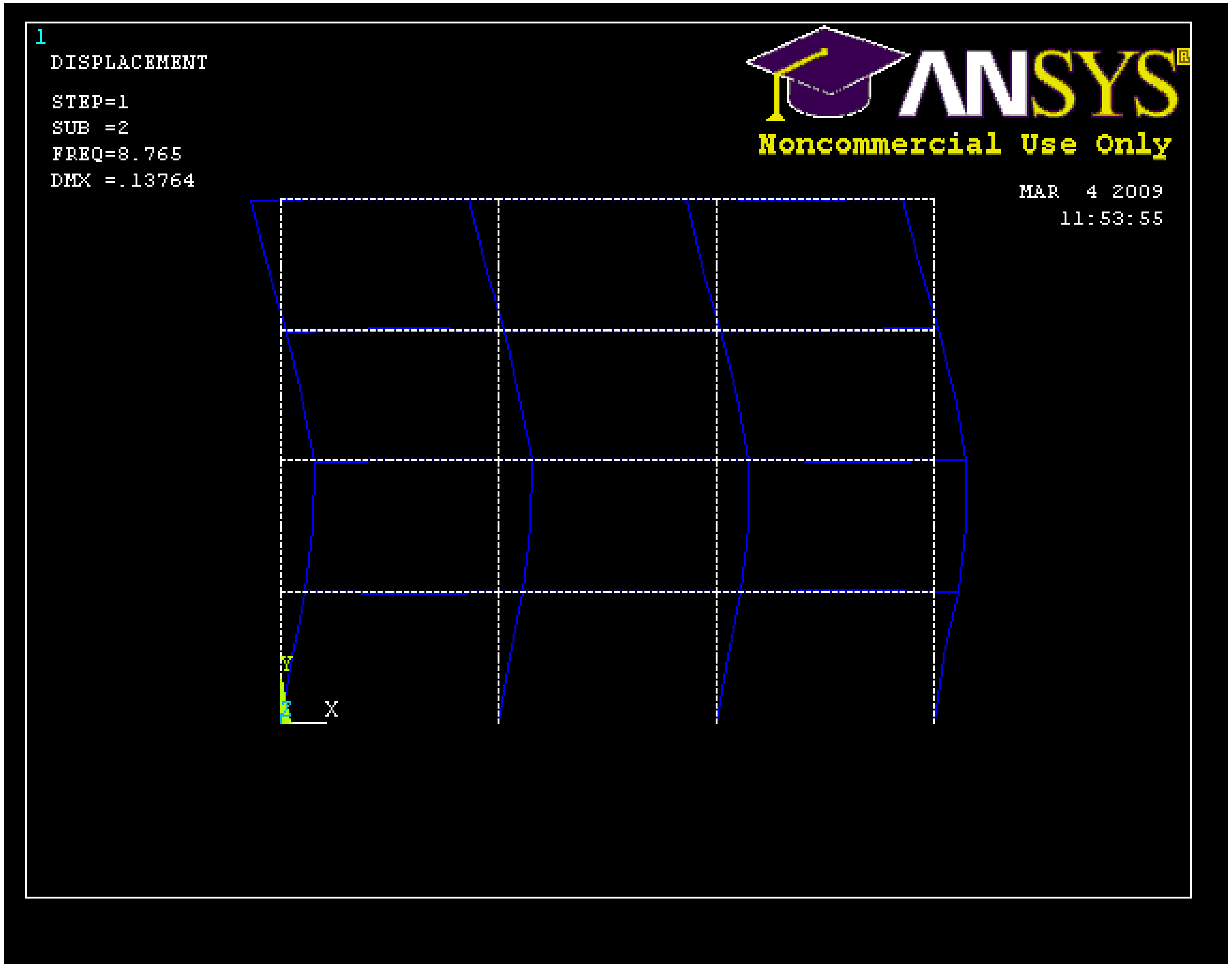
Mode shapes of a four storey 2D frame

- First mode shape



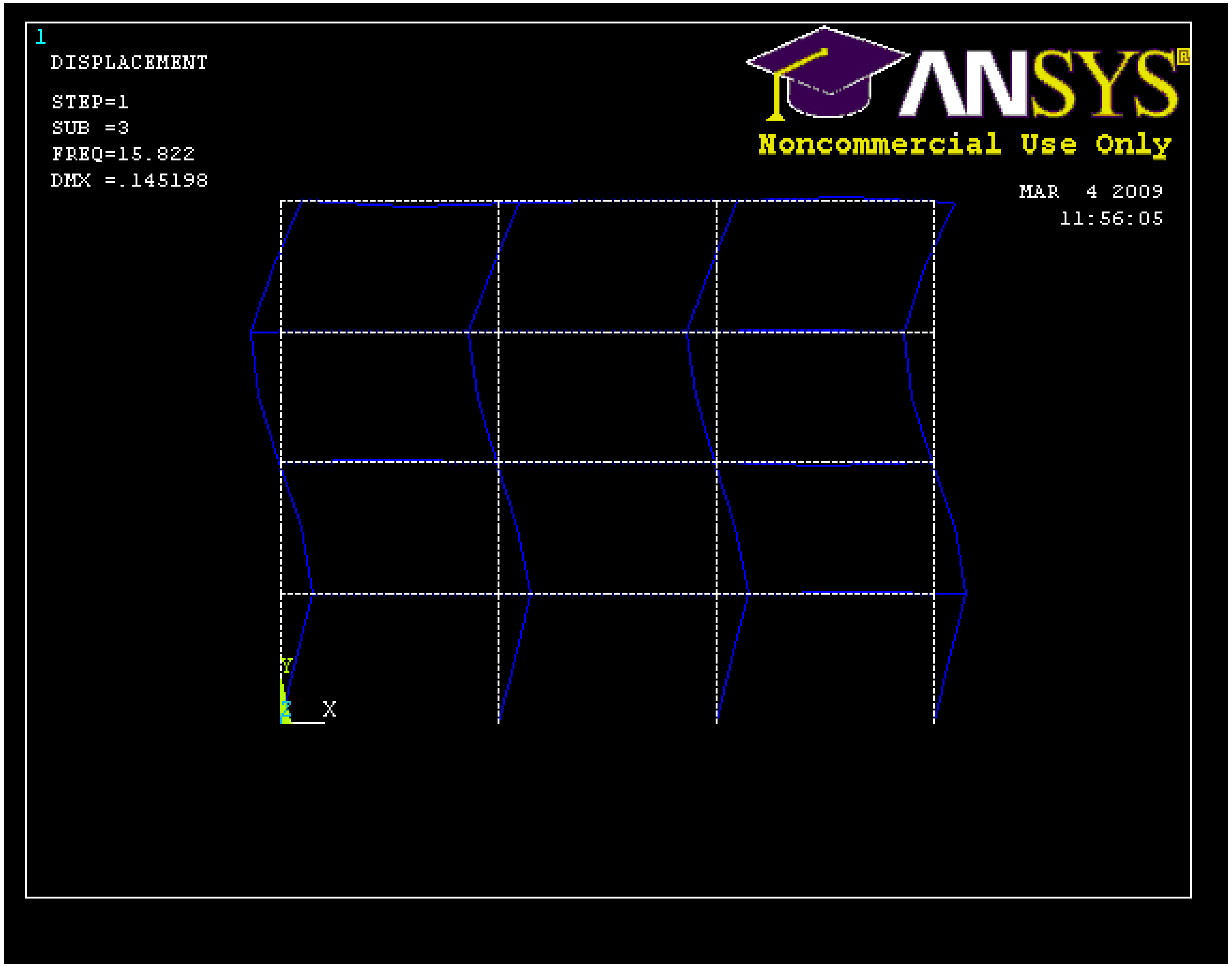
Mode shapes of a four storey 2D frame

- Second mode shape



Mode shapes of a four storey 2D frame

- Third mode shape



Undamped free vibrations

Example:

Determine the eigenvalues and eigenvectors of a vibrating system for which

$$[m] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{and} \quad [k] = \begin{bmatrix} 1 & -2 & 1 \\ -2 & 4 & -2 \\ 1 & -2 & 1 \end{bmatrix}$$

Given: Mass and stiffness matrices.

Find: Eigenvalues and eigenvectors.

Example

Solution: The eigenvalue equation $[[k] - \lambda[m]]\,\vec{X} = \vec{0}$ can be written in the form

$$\begin{bmatrix} (1-\lambda) & -2 & 1 \\ -2 & 2(2-\lambda) & -2 \\ 1 & -2 & (1-\lambda) \end{bmatrix} \begin{Bmatrix} X_1 \\ X_2 \\ X_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}$$

where $\lambda = \omega^2$. The characteristic equation gives

$$\left|[k] - \lambda[m]\right| = \lambda^2 (\lambda - 4) = 0$$

so

$$\lambda_1 = 0,\ \lambda_2 = 0,\ \lambda_3 = 4$$

Eigenvector for $\lambda_3 = 4$: Using $\lambda_3 = 4$, Eq. (E.1) gives

$$-3\,X_1^{(3)} - 2\,X_2^{(3)} + X_3^{(3)} = 0$$

$$-2\,X_1^{(3)} - 4\,X_2^{(3)} - 2\,X_3^{(3)} = 0$$

$$X_1^{(3)} - 2\,X_2^{(3)} - 3\,X_3^{(3)} = 0$$

If $X_1^{(3)}$ is set equal to 1, Eqs. (E.3) give the eigenvector $\vec{X}^{(3)}$:

$$\vec{X}^{(3)} = \begin{Bmatrix} 1 \\ -1 \\ 1 \end{Bmatrix}$$

Example

Solution:

When the characteristic equation possesses repeated roots, the corresponding mode shapes are not unique.

Eigenvector for $\lambda_1 = \lambda_2 = 0$*:* The value $\lambda_1 = 0$ or $\lambda_2 = 0$ indicates that the system is degenerate (see Section 6.12). Using $\lambda_1 = 0$ in Eq. (E.1), we obtain

$$\begin{aligned} X_1^{(1)} - 2\,X_2^{(1)} + X_3^{(1)} &= 0 \\ -2\,X_1^{(1)} + 4\,X_2^{(1)} - 2\,X_3^{(1)} &= 0 \\ X_1^{(1)} - 2\,X_2^{(1)} + X_3^{(1)} &= 0 \end{aligned} \tag{E.5}$$

Example

Solution:

All these equations are of the form

$$X_1^{(1)} = 2\, X_2^{(1)} - X_3^{(1)}$$

Thus the eigenvector corresponding to $\lambda_1 = \lambda_2 = 0$ can be written

$$\vec{X}^{(1)} = \begin{Bmatrix} 2\, X_2^{(1)} - X_3^{(1)} \\ X_2^{(1)} \\ X_3^{(1)} \end{Bmatrix}$$

If we choose $X_2^{(1)} = 1$ and $X_3^{(1)} = 1$, we obtain

$$\vec{X}^{(1)} = \begin{Bmatrix} 1 \\ 1 \\ 1 \end{Bmatrix}$$

If we select $X_2^{(1)} = 1$ and $X_3^{(1)} = -1$, Eq. (E.6) gives

$$\vec{X}^{(1)} = \begin{Bmatrix} 3 \\ 1 \\ -1 \end{Bmatrix}$$

Rigid body motion

- An unrestrained system is one that has no restraints or supports and that can move as a rigid body. It is not uncommon to see in practice systems that are not attached to any stationary frame.
- Such systems are capable of moving as rigid bodies, which can be considered as modes of oscillation with zero frequency.
- A semidefinite system such as this, has a singular stiffness matrix. In systems that are not properly restrained, rigid-body displacements can take place without the application of any force. Thus, denoting a possible rigid-body displacement by $\mathbf{u}_r$, we have

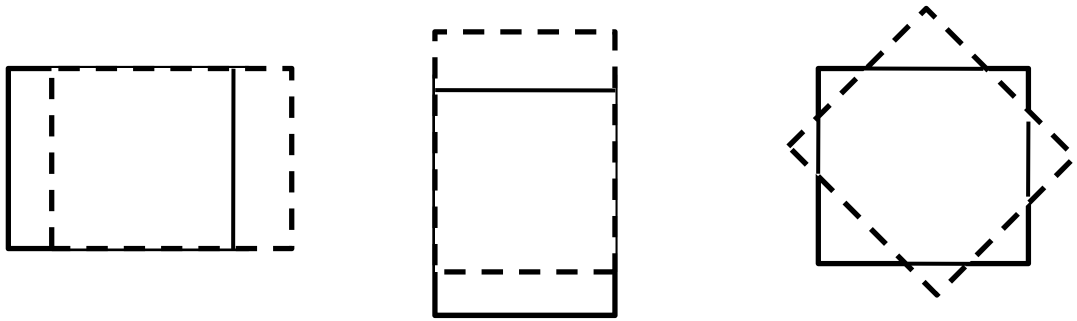
$$\mathbf{f}_r = \mathbf{K}\mathbf{u}_r = \mathbf{0}$$

- For a nonzero $\mathbf{u}_r$, the above equation can be satisfied provided only that **K** is singular. In this case, the below equation can only be satisfied when ω=0.

$$\left[\mathbf{K} - \omega^2\mathbf{M}\right]\mathbf{u}_r = \mathbf{0}$$

Rigid body motion

- The rigid body displacements are those displacement modes that the element must be able to undergo as a rigid body without stresses being developed in it.
- Rigid body displacement shapes are also referred to as rigid body modes.
- A system can, of course, have more than one rigid body mode. In the most general case, up to six rigid body modes are possible. For example, a spacecraft or an aeroplane in flight has all six possible rigid-body modes, three translations and three rotations, one along each of the three axis.



Rigid body modes of a plane stress element

Orthogonality of modes

- The natural modes corresponding to different natural frequencies can be shown to satisfy the following orthogonality conditions. When $\omega_n \neq \omega_r$:

$$\phi_n^T \mathbf{k} \phi_r = 0 \qquad \phi_n^T \mathbf{m} \phi_r = 0$$

- **Proof:** The nth natural frequency and mode satisfy

$$\mathbf{k}\phi_n = \omega_n^2 \mathbf{m}\phi_n$$

Premultiplying the above equation by ϕ_r^T

$$\phi_r^T \mathbf{k}\phi_n = \omega_n^2 \phi_r^T \mathbf{m}\phi_n$$

Similarly the rth natural frequency and mode shape satisfy

$$\mathbf{k}\phi_r = \omega_r^2 \mathbf{m}\phi_r$$

Orthogonality of modes

Premultiplying $\mathbf{k}\phi_r = \omega_r^2 \mathbf{m}\phi_r$ by ϕ_n^T gives:

$$\phi_n^T \mathbf{k}\phi_r = \omega_r^2 \phi_n^T \mathbf{m}\phi_r$$

The transpose of the matrix on the left side of $\phi_r^T \mathbf{k}\phi_n = \omega_n^2 \phi_r^T \mathbf{m}\phi_n$ will equal the transpose of the matrix on the right side of the equation:

$$\phi_n^T \mathbf{k}\phi_r = \omega_n^2 \phi_n^T \mathbf{m}\phi_r$$

Subtracting the first equation from the second equation:

$$\left(\omega_n^2 - \omega_r^2\right)\phi_n^T \mathbf{m}\phi_r = 0$$

The equation $\phi_n^T \mathbf{m}\phi_r = 0$ is true when $\omega_n \neq \omega_r$ which for systems with positive natural frequencies implies that $\omega_n \neq \omega_r$

Modal equations for undamped systems

- The equations of motion for a linear MDOF system without damping is:

$$\mathbf{m}\ddot{\mathbf{x}} + \mathbf{k}\mathbf{x} = \mathbf{p}(t)$$

- The simultaneous solution of these coupled equations of motion that we have illustrated before for a 2 dof system subjected to harmonic excitation is not efficient for systems with more DOF, nor is it feasible for systems excited by other types of forces. Consequently, it is advantegous to transform these equations to modal coordinates.
- The displacement vector **x** of a MDOF system can be expanded in terms of modal contributions. Thus, the dynamic response of a system can be expressed as:

$$\mathbf{x}(t) = \sum_{r=1}^{N} \phi_r q_r(t) = \boldsymbol{\varphi}\mathbf{q}(t)$$

Modal equations for undamped systems

- Using the equation $\mathbf{x}(t) = \sum_{r=1}^{N} \phi_r q_r(t) = \boldsymbol{\varphi}\mathbf{q}(t)$, the coupled equations in $x_j(t)$ given below

$$\mathbf{m}\ddot{\mathbf{x}} + \mathbf{k}\mathbf{x} = \mathbf{p}(t)$$

can be transformed to a set of uncoupled equations with modal coordinates $q_n(t)$ as the unknowns. Substituting the first equation into the second:

$$\sum_{r=1}^{N} \mathbf{m}\phi_r \ddot{q}_r(t) + \sum_{r=1}^{N} \mathbf{k}\phi_r q_r(t) = \mathbf{p}(t)$$

Premultiplying each term in this equation by ϕ_n^T gives:

$$\sum_{r=1}^{N} \phi_n^T \mathbf{m}\phi_r \ddot{q}_r(t) + \sum_{r=1}^{N} \phi_n^T \mathbf{k}\phi_r q_r(t) = \phi_n^T \mathbf{p}(t)$$

Modal equations for undamped systems

- Because of the orthogonality relations $\phi_n^T \mathbf{k} \phi_r = 0$ $\phi_n^T \mathbf{m} \phi_r = 0$, all terms in each of the summations vanish except the r=n term, reducing the equation to:

$$\left(\phi_n^T \mathbf{m} \phi_n\right)\ddot{q}_n(t) + \left(\phi_n^T \mathbf{k} \phi_n\right)q_n(t) = \phi_n^T \mathbf{p}(t)$$

or

$$M_n \ddot{q}_n(t) + K_n q_n(t) = P_n(t)$$

where

$$M_n = \phi_n^T \mathbf{m} \phi_n \quad K_n = \phi_n^T \mathbf{k} \phi_n \quad P_n(t) = \phi_n^T \mathbf{p}(t)$$

- The above equation may be interpreted as the equation governing the response $q_n(t)$ of the SDOF system with mass M_n, stiffness K_n, and exciting force $P_n(t)$.
- Therefore M_n is called the generalized mass for the nth natural mode, K_n the generalized stiffness for the nth mode, and $P_n(t)$ the generalized force for the nth mode. These parameters only depend on the nth mode.

Modal equations for damped systems

- When damping is included, the equations of motion for a MDOF system are:

$$\mathbf{m}\ddot{\mathbf{x}} + \mathbf{c}\dot{\mathbf{x}} + \mathbf{k}\mathbf{x} = \mathbf{p}(\mathbf{t})$$

- Using the transformation

$$\mathbf{x}(t) = \sum_{r=1}^{N} \phi_r q_r(t) = \boldsymbol{\varphi}\mathbf{q}(t)$$

where ϕ_r are the natural modes of the system without damping, these equations can be written in terms of the modal coordinates. Unlike the case of undamped systems, these modal equations may be coupled through the damping terms. However, for certain forms of damping that are reasonable idealizations for many structures, the equations become uncoupled, just as for undamped systems. Substituting the second equation into the first, we obtain:

$$\sum_{r=1}^{N} \mathbf{m}\phi_r \ddot{q}_r(t) + \sum_{r=1}^{N} \mathbf{c}\phi_r \dot{q}_r(t) + \sum_{r=1}^{N} \mathbf{k}\phi_r q_r(t) = \mathbf{p}(\mathbf{t})$$

Modal equations for damped systems

- Premultiplying each term in this equation by ϕ_n^T gives:

$$\sum_{r=1}^{N} \phi_n^T \mathbf{m} \phi_r \ddot{q}_r(t) + \sum_{r=1}^{N} \phi_n^T \mathbf{c} \phi_r \dot{q}_r(t) + \sum_{r=1}^{N} \phi_n^T \mathbf{k} \phi_r q_r(t) = \phi_n^T \mathbf{p}(t)$$

which can be rewritten as:

$$M_n \ddot{q}_n(t) + \sum_{r=1}^{N} C_{nr} \dot{q}_r(t) + K_n q_n(t) = P_n(t)$$

where

$$C_{nr} = \phi_n^T \mathbf{c} \phi_r$$

The above N equations can be written in matrix form as:

$$\mathbf{M}\ddot{\mathbf{q}} + \mathbf{C}\dot{\mathbf{q}} + \mathbf{K}\mathbf{q} = \mathbf{P}(t)$$

Here C is a nondiagonal matrix of coefficients C_{nr}.

Modal equations for damped systems

- The modal equations will be uncoupled if the system has classical damping. For such systems $C_{nr}=0$ if $n \neq r$ and C_n can be expressed as:

$$C_n = 2\zeta_n M_n \omega_n$$

- For such systems:

$$M_n \ddot{q}_n + C_n \dot{q}_n + K_n q_n = P_n(t)$$

- Dividing by M_n:

$$\ddot{q}_n + 2\zeta_n \omega_n \dot{q}_n + \omega_n^2 q_n = \frac{P_n(t)}{M_n}$$

where ζ_n is the damping ratio for the nth mode.