GRAPH THEORY and APPLICATIONS

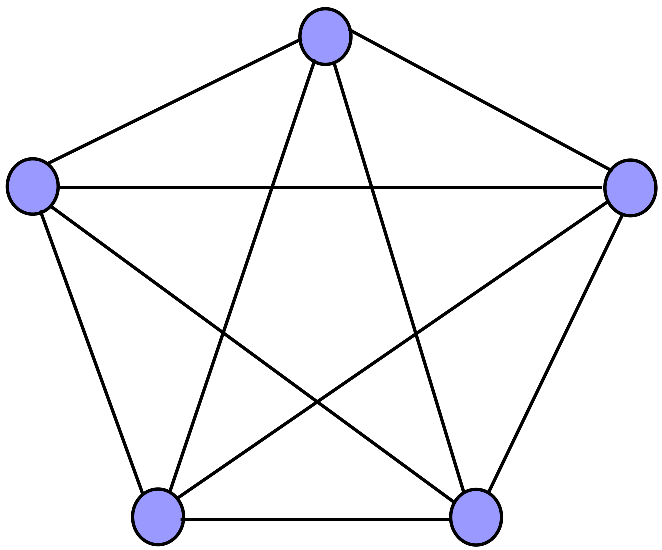
Planar Graphs

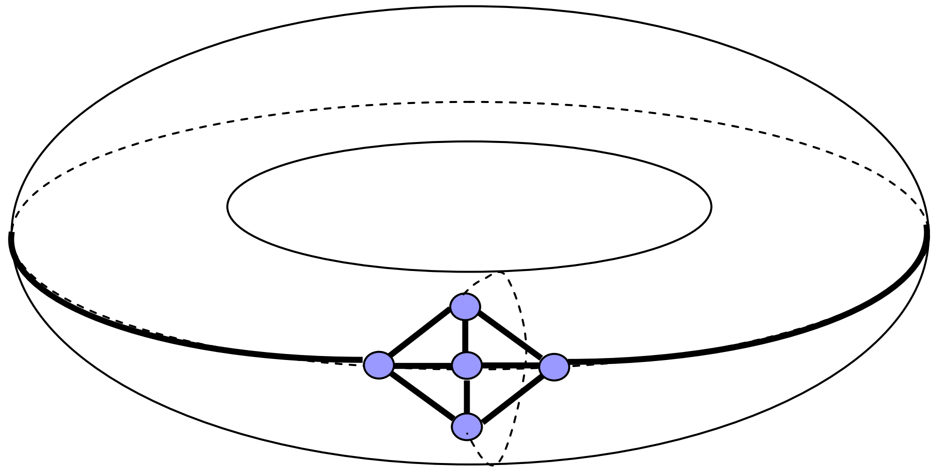
Planar Graph

- A graph is planar if it can be drawn on a plane surface with no two edges intersecting.
- G is said to be *embedded* in the plane.
- We can extend the idea of embedding, to other surfaces.
- K_5 cannot be embedded on a plane, but it can be embedded on a toroidal surface.

Theorem: A graph G is embeddable in the plane iff it is embeddable on the sphere.

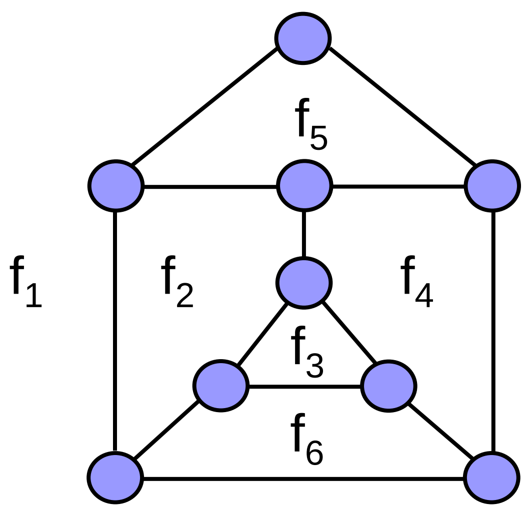
Example: K_5

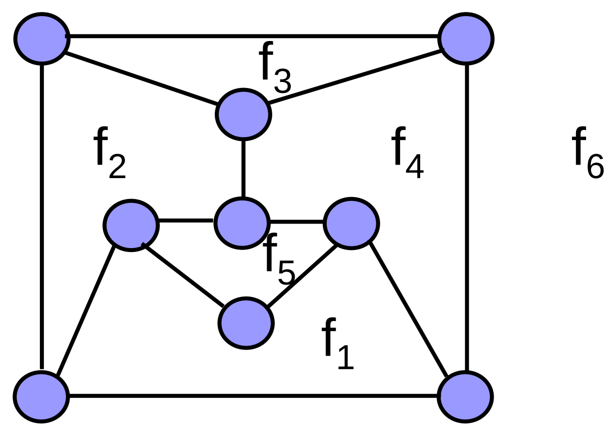




Faces (regions)

- A planar representation of a graph divides the plane into a number of connected regions: faces.
- Each face is bounded by edges.
- One of the faces encloses the graph: exterior face.





Euler's formula

Theorem: A planar embedding of a graph can be transformed into a different planar embedding such that any specified face becomes the exterior face.

- There is a simple formula connecting the number of faces, edges, and vertices in a connected planar graph: Euler's formula.

Theorem: If G is a connected, planar graph, then:

$$n - |E| + f = 2$$

Degree of a face

- Degree of a face, d(f): Number of edges bounding the face.

Lemma: For a simple, planar graph G, we have:

$$2|E| = \sum_i d(f_i) = \sum_i i \cdot n(i)$$

- Each edge contributes one to the degree of each of two faces it separates.

$n(i)$: number of vertices of degree i

Corollaries to Euler's formula

Corollary 1: For any simple, connected, planar graph G, with $|E| > 2$, the following holds:

$$|E| \leq 3n - 6$$

Proof:

- Each face is bounded by at least 3 edges, so:

$$\sum_i d(f_i) \geq 3f$$

- Substitute $3f$ with $6 - 3n + 3|E|$, and use the lemma.

Corollaries to Euler's formula

Corollary 2: For any simple connected *bipartite* planar graph G, with $|E| > 2$, the following holds:

$$|E| \leq 2n - 4$$

Proof:

- Each face of G is bounded by at least 4 edges.
- The result then follows as for the previous corollary.

Corollaries to Euler's formula

Corollary 3: In a simple, connected, planar graph there exists at least one vertex of degree at most 5.

Proof:

- From first corollary: $|E| \leq 3n - 6$
- Also: $n = \sum_i n(i)$ and $2|E| = \sum_i i \cdot n(i)$
- By substitution:

$$\sum_i (6 - i) \cdot n(i) \geq 12$$

- Left-hand size must be positive. i and $n(i)$ are always nonnegative.

Nonplanarity of K_5 and $K_{3,3}$

- K_5 cannot be planar:
  - It has 5 vertices and 10 edges.
  - Inequality of corollary 1 is violated.

    $|E| \leq 3n - 6 \Rightarrow 10 \not\leq 3*5 - 6$
- $K_{3,3}$ cannot be planar:
  - It has 6 vertices and 9 edges.
  - Inequality of corollary 2 is not satisfied.

    $|E| \leq 2n - 4 \Rightarrow 9 \not\leq 2*6 - 4$

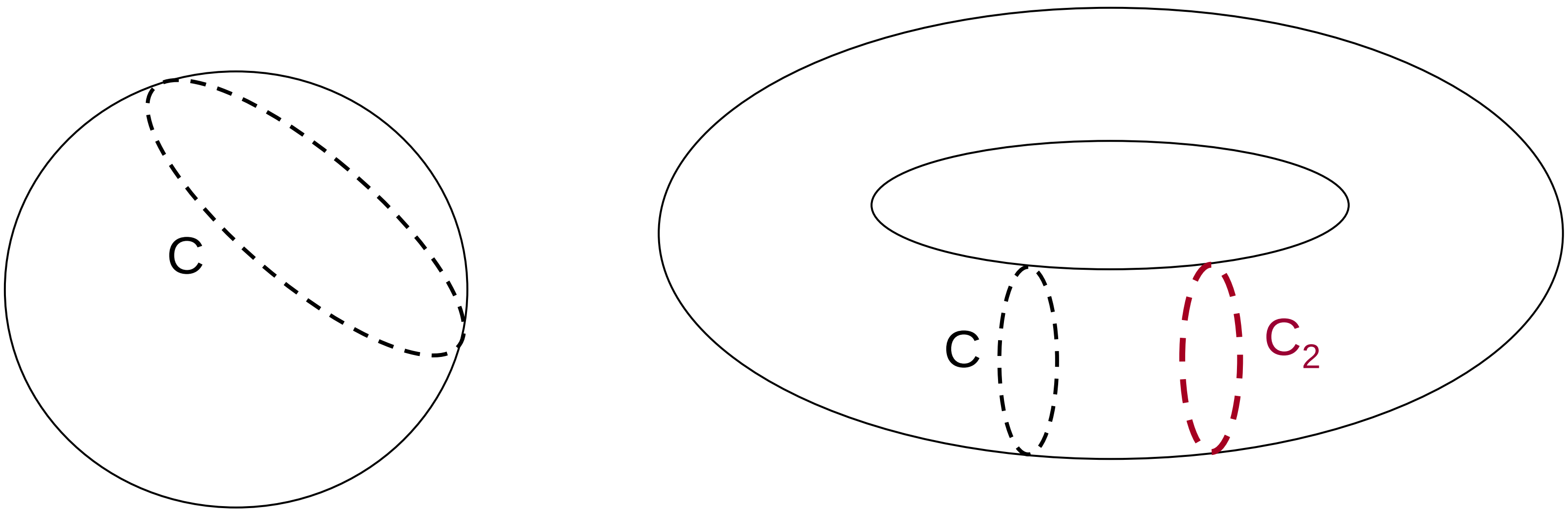
- All three corollaries are necessary, but not sufficient conditions.

Sphere vs. torus

- K_5 and $K_{3,3}$ are toroidal graphs, i.e., they can be embedded on the surface of a torus.
- Sphere and torus are topologically different.
  - Any single closed line (curve) embedded on a spherical surface divides the surface into two regions.
  - A closed curve embedded on a toroidal surface will not necessarily divide it into two regions.
  - 2 non-intersecting closed curves are guaranteed to divide the surface of a torus.

Sphere vs. torus

- Example:



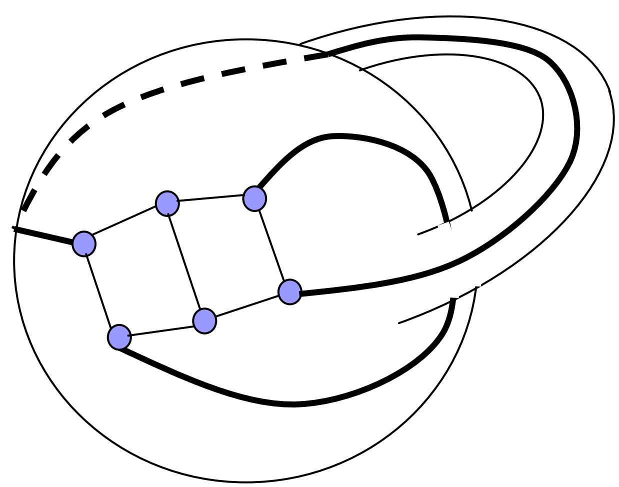
Genus

- For a nonnegative integer g, we can construct a surface in which:
  - it is possible to embed g non-intersecting closed curves
  - without separating the surface into two regions.
- If for some surface, (g+1) closed curves <u>*always*</u> cause a separation, then the surface has a ***genus*** g.
- Spherical surfaces have genus g = 0
- Toroidal surfaces have genus g = 1

Genus

- The genus is a topological property of a surface, and remains the same if the surface is deformed.
- The toroidal surface:
  - Similar to spherical surface with a *handle*.

$K_{3,3}$ embedded on a toroidal surface



Crossing number

- Any surface of genus g is topologically equivalent to a spherical surface with g *handles*.
- Graph of genus g:
  - A graph that can be embedded on a surface of genus g
  - but not on a surface of genus g – 1.
- Crossing number of a graph: Minimum number of crossings of edges for the graph drawn on the plane.
  - Genus of a graph will not exceed its crossing number.

A theorem

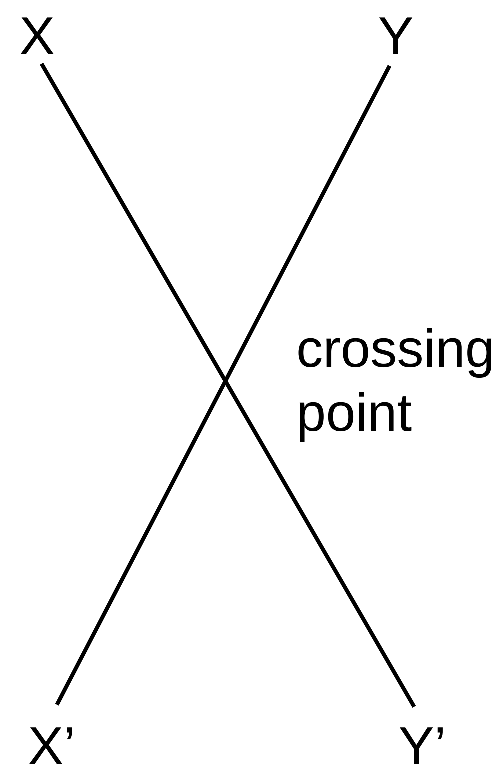
Theorem: If G is a connected graph with genus g, n vertices, e edges, and embedding of G has f faces, then:

$$f = e - n + 2 - 2g$$

- For g = 0:
  - This theorem becomes Euler's formula.
- Handles connect two distinct faces of the surface.

An application: Electrical circuits

- Genus and crossing number have importance in the manufacture of electrical circuits on planar sheets.
- A convenient method:
  - Divide the circuit into planar subcircuits
  - Separate them with insulating sheets
  - Make connections between subcircuits, at the vertices of the graph.

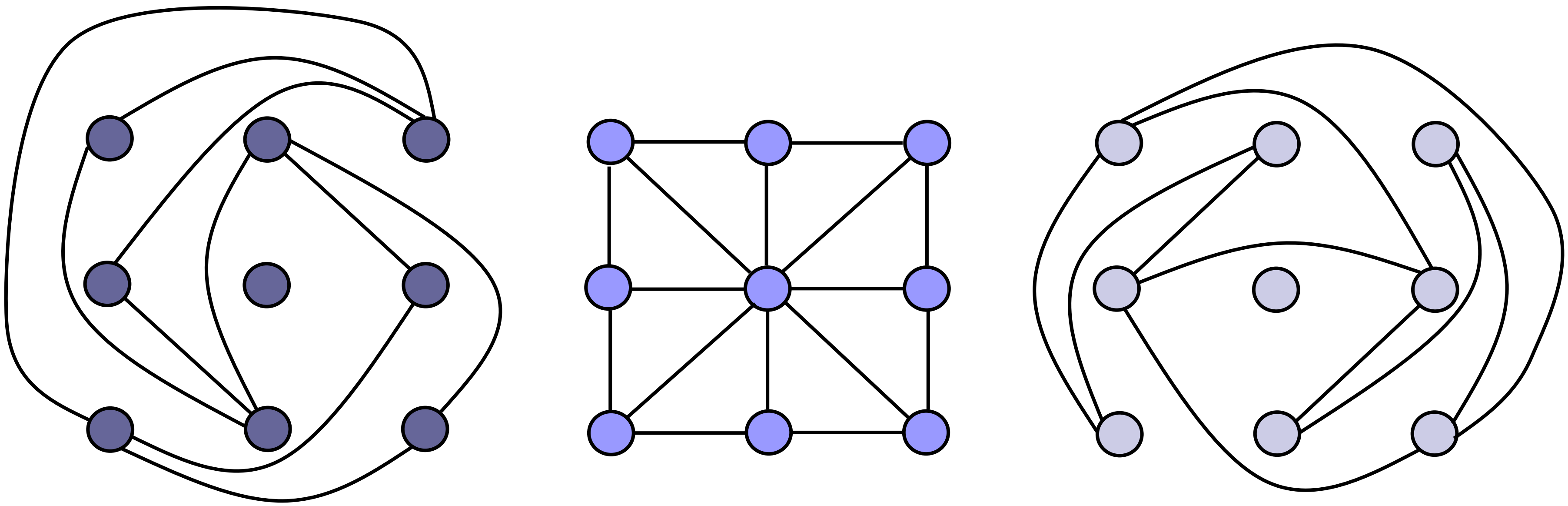


Thickness

- The problem of separating the electrical circuit sheets into planar subcircuits, is equivalent to decomposing the associated graph into planar subgraphs.
- The **thickness** of a graph: T(G)
  The minimum number of planar subgraphs of G whose *union* is G.
- Union of $G_1(V,E_1)$ and $G_2(V,E_2)$ is the graph $(V,E_1 \cup E_2)$

Example

Three graphs whose union is K_9



Corollaries

Corollary: The thickness T of a simple graph with n vertices and e edges satisfies:

$$T \geq \left\lceil \frac{e}{3n-6} \right\rceil$$

Corollary: The genus g of a simple graph with $n \geq 4$ vertices and e edges satisfies:

$$g \geq \left\lceil \frac{1}{6}(e-3n)+1 \right\rceil$$

Special cases

- Specific results for thickness and genus are known for special graphs (complete, bipartite,…)
- In complete graphs:

$$e = \frac{1}{2} n \cdot (n-1)$$

  - The corollaries give:

$$g \geq \left\lceil \frac{1}{12} (n-3) \cdot (n-4) \right\rceil$$

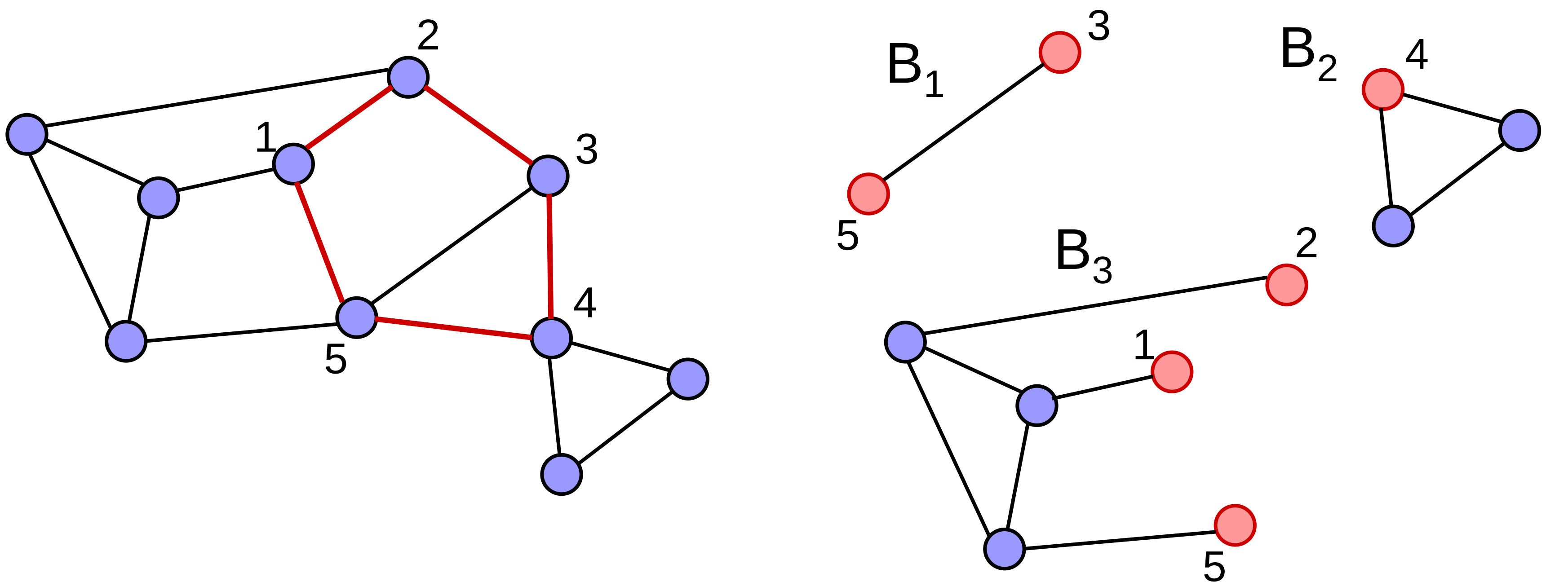
$$T \geq \left\lceil \frac{n \cdot (n-1)}{6(n-2)} \right\rceil = \left\lfloor \frac{n \cdot (n-1) + (6n-14)}{6(n-2)} \right\rfloor = \left\lfloor \frac{1}{6} (n+7) \right\rfloor$$

Characterization of planarity

- Let $G_1(V_1,E_1)$ be a subgraph of $G(V,E)$.
- A piece of G relative to G_1 is, either:
  - an edge $(u,v) \in E$, where
    - $(u,v) \notin E_1$,and
    - $u,v \in V_1$

  or:
  - a connected component of $(G - G_1)$ plus any edges incident with this component.
- For any piece B, the vertices which B has in common with G_1 are called the points of contact of B.

Bridge



- If a piece has two or more points of contact then it is called a bridge.
  - B_1 and B_3 are bridges, B_2 is not.

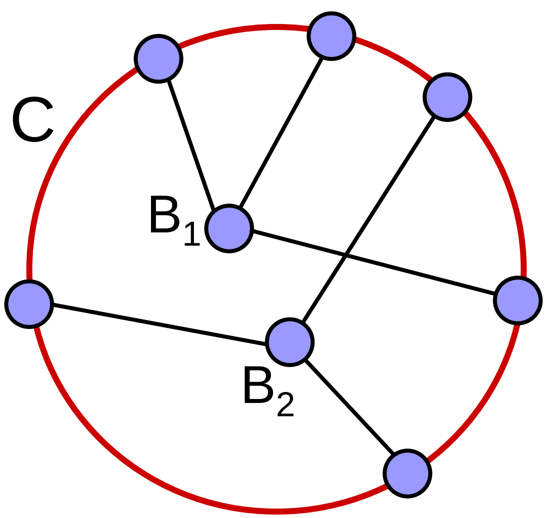
Bridges of a block

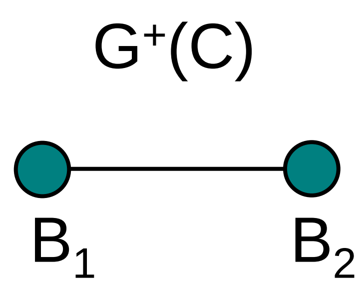
- A graph is planar iff each of its blocks is planar.
- Thus, in question of planarity, we are dealing with blocks.
- Any piece of a block with respect to any proper subgraph is a bridge.

- Let C be a circuit which is a subgraph of G.
- C divides the plane into two faces:
  - an interior face, and
  - an exterior face.

Incompatible bridges

- Two bridges B_1 and B_2 are incompatible ($B_1 \not\approx B_2$), if at least two of their edges cross, when placed in the same face of the plane defined by C.



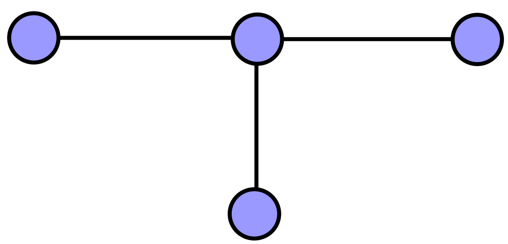


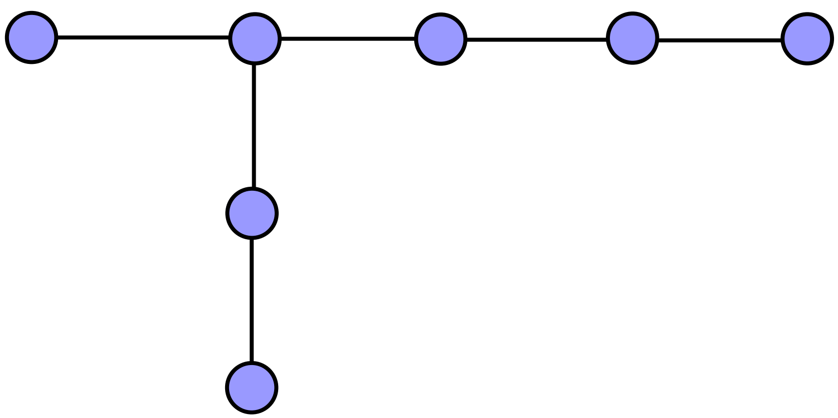
Incompatible bridges

- An auxiliary graph $G^+(C)$ relative to circuit C has
  - a vertex set consisting of a vertex for each bridge
  - an edge between any two vertices B_i and B_j iff $B_i \not\approx B_j$.
- Suppose $G^+(C)$ is a bipartite graph with bipartition (B,B').
  - The bridges in B may be embedded in one face of C, and
  - the bridges in B' may be embedded in the other face.

Homeomorphism

- Two graphs are homeomorphic if one can be made isomorphic to the other by the addition or deletion of vertices of degree two.





Kuratowski's theorem

- Kuratowski's theorem:

Theorem: A graph is planar iff it has no subgraph homeomorphic to K_5 and $K_{3,3}$.

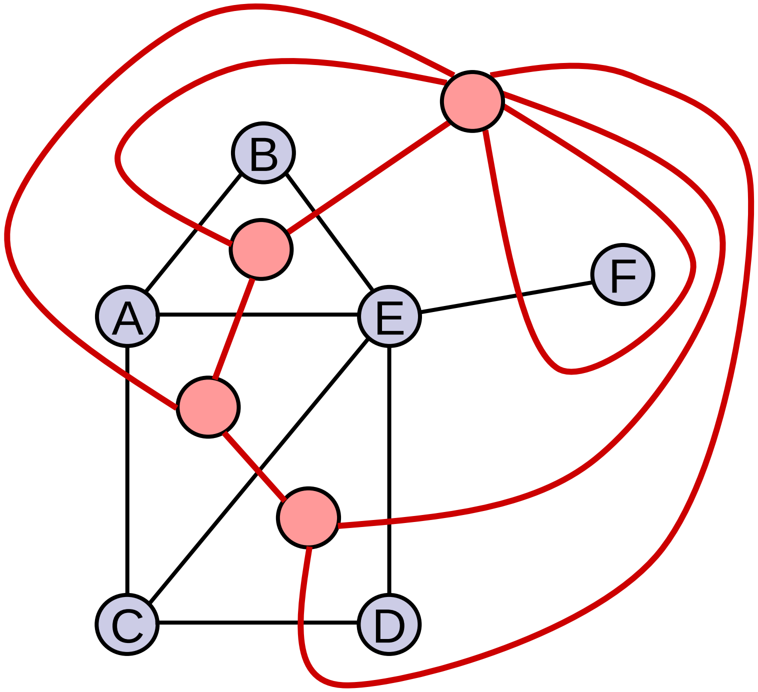
- A more appropriate insight into the planarity is as follows:

Theorem: A graph is planar iff for every circuit C of G the auxiliary graph $G^+(C)$ is bipartite.

Dual graphs

- Given a planar representation G^p of a graph, the construction rules of its dual G*:
  - A vertex of G* is associated with each face of G^p
  - For each edge e_i of G^p there is an assoiciated edge e_i* of G*.
  - If e_i separates the faces f_j and f_k in G^p, then e_i* connects the two vertices of G* associated with f_j and f_k .
- The dual graph is also planar.

Example

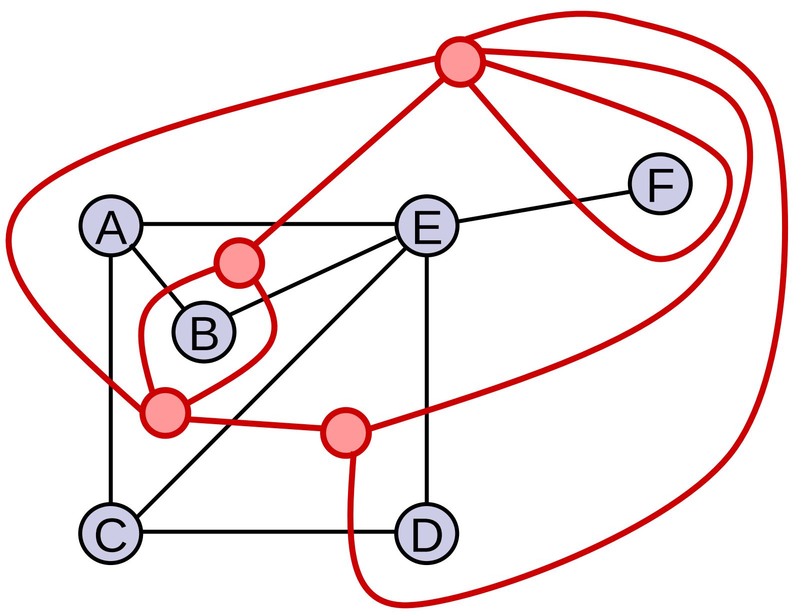


- Either graph is the dual of the other.

Different dual graphs

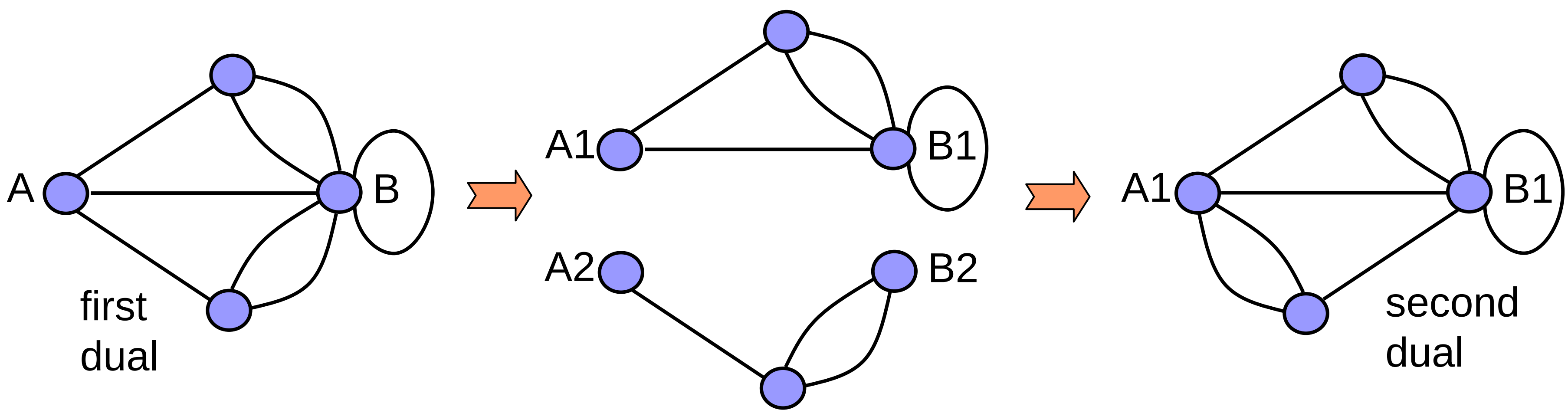
The dual of a planar representation of G, not the dual of G.

Another planar representation of the example:



Duals

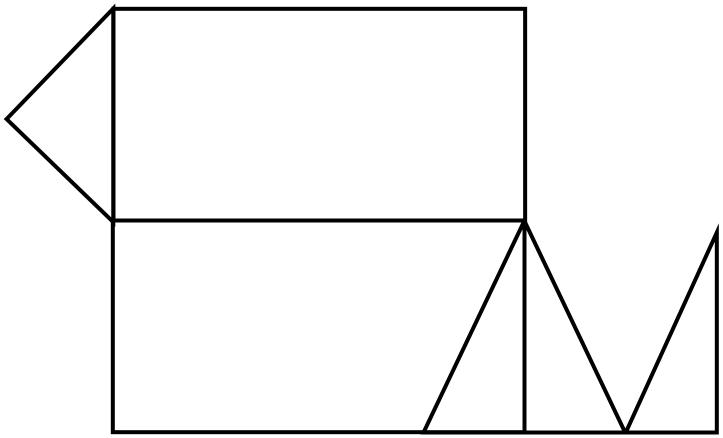
- In the previous example, we see that the duals of the two planar representations are not isomorphic.
- There is a constructional relationship between the duals of different planar representations of a graph.

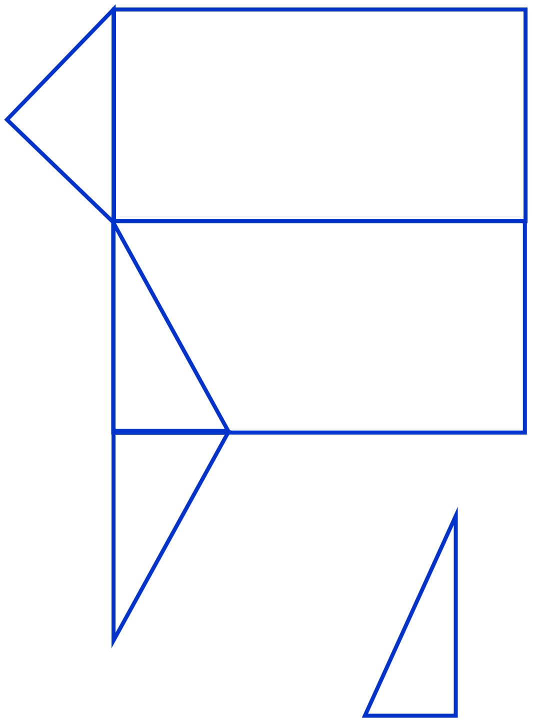


2-isomorphic graphs

- Any two graphs G_1 and G_2 are 2-isomorphic if they become isomorphic under repeated application of the following operations:
  - Separation of G_1 or G_2 into two or more components at cut-points
  - If G_1 and G_2 can be:
    - divided into two disjoint subgraphs with two vertices in common,
    - then separate at these vertices A and B,
    - and reconnect so that A1 coincides with B2, and A2 coincides with B1

Example





These two graphs are 2-isomorphic.

Theorems

Theorem: All the duals of a planar graph G are 2-isomorphic; any graph 2-isomorphic to a dual of G is also a dual of G.

Theorem: Every planar graph has a dual.

Theorem: A graph has a dual iff it is planar.

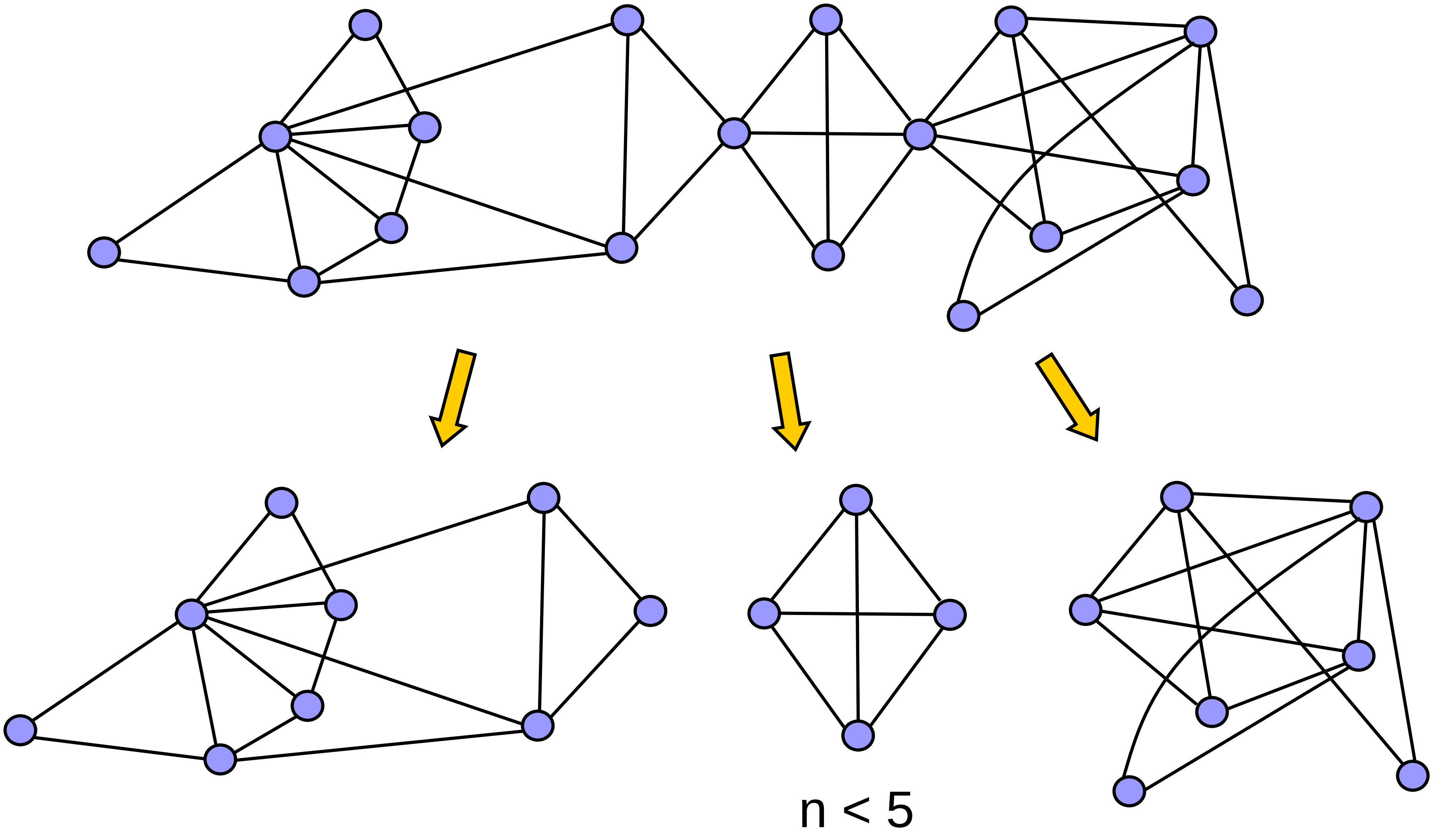
Testing the planarity

- Some preprocessing to simplify the task:
  - If the graph is not connected, then consider each component separately.
  - If the graph has cut-vertices, then it is planar iff each of its blocks is planar. Therefore, test each block separately.
  - Loops may be removed.
  - Parallel edge may be removed.
  - Each vertex of degree 2 plus its incident edges can be replaced by a single edge.
- These steps may be applied repeatedly and alternatively until neither can be applied further.

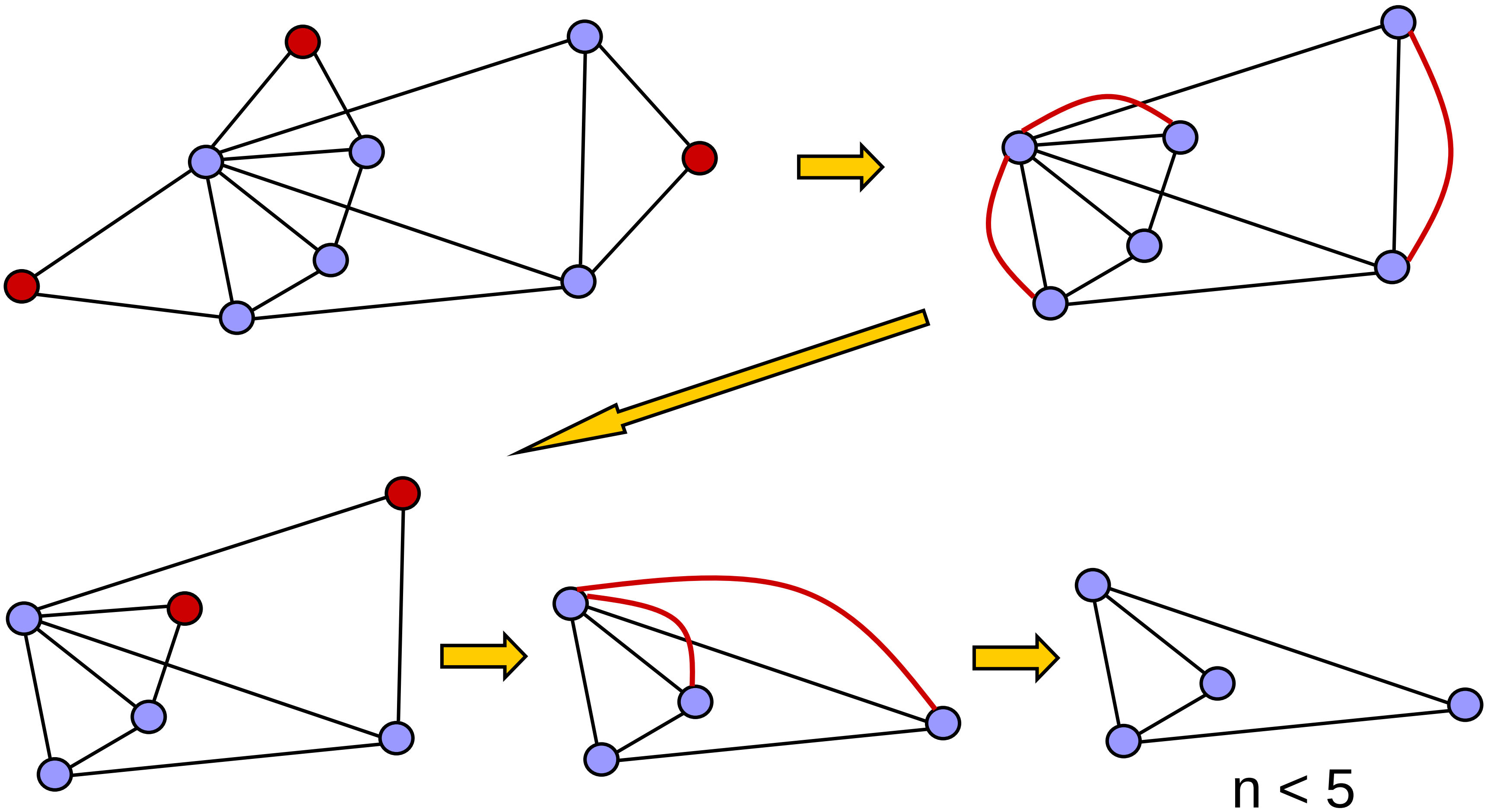
Simple tests

- Following the simplifications, two elementary tests can be applied:
  - If $e < 9$ or $n < 5$ then the graph must be planar.
  - If $e > 3n - 6$ then the graph must be non-planar.
- If these tests fail to resolve the question of planarity, then we need to use a more elaborate test.

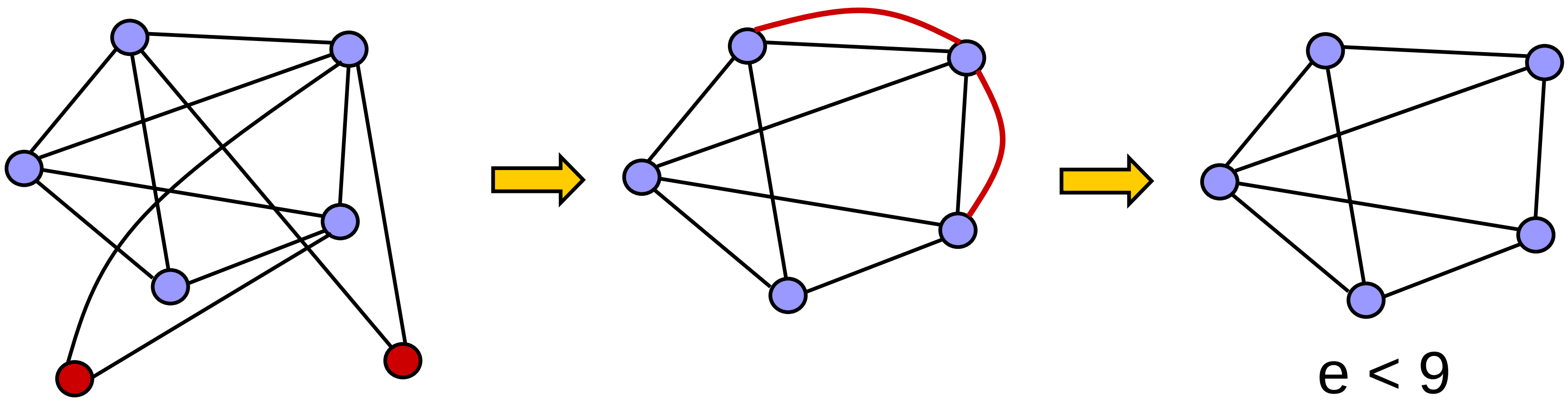
Example



Example



Example

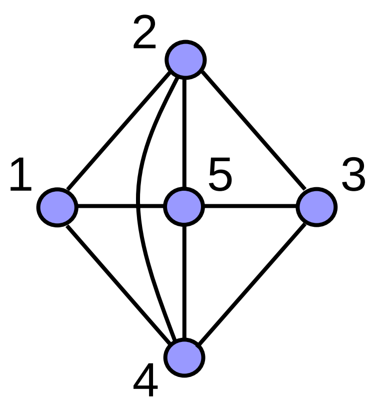


Planarity test algorithms

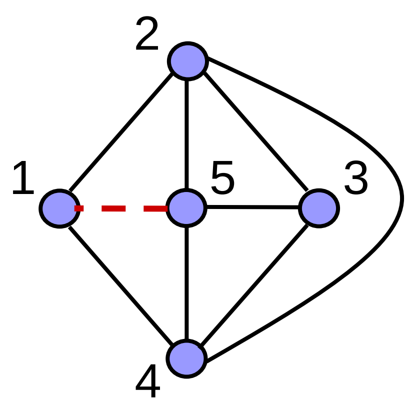
- Many algorithms have been published.
  - Demoucron, Malgrange, and Pertuiset (1964)
  - Lempel, Even, and Cederbaum (1967)
  - Even and Tarjan (1976)
  - Leuker and Booth (1976)
- The last algorithm is simpler, and fairly efficient.

Admissable subgraph

- Let H' be a planar embedding of the subgraph H of G.
  - If there is a planar embedding G' such that H' $\subseteq$ G' then,
  - H' is said to be G-admissable.

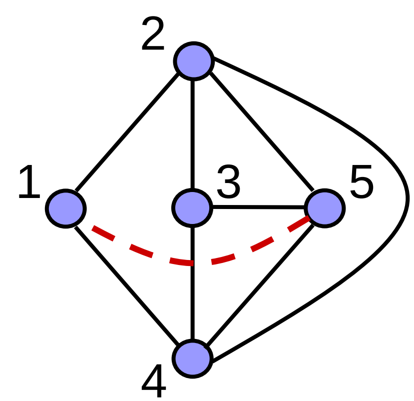


G



G-admissable

H = G – (1,5)



G-inadmissable

Planarity testing algorithm

Notations:

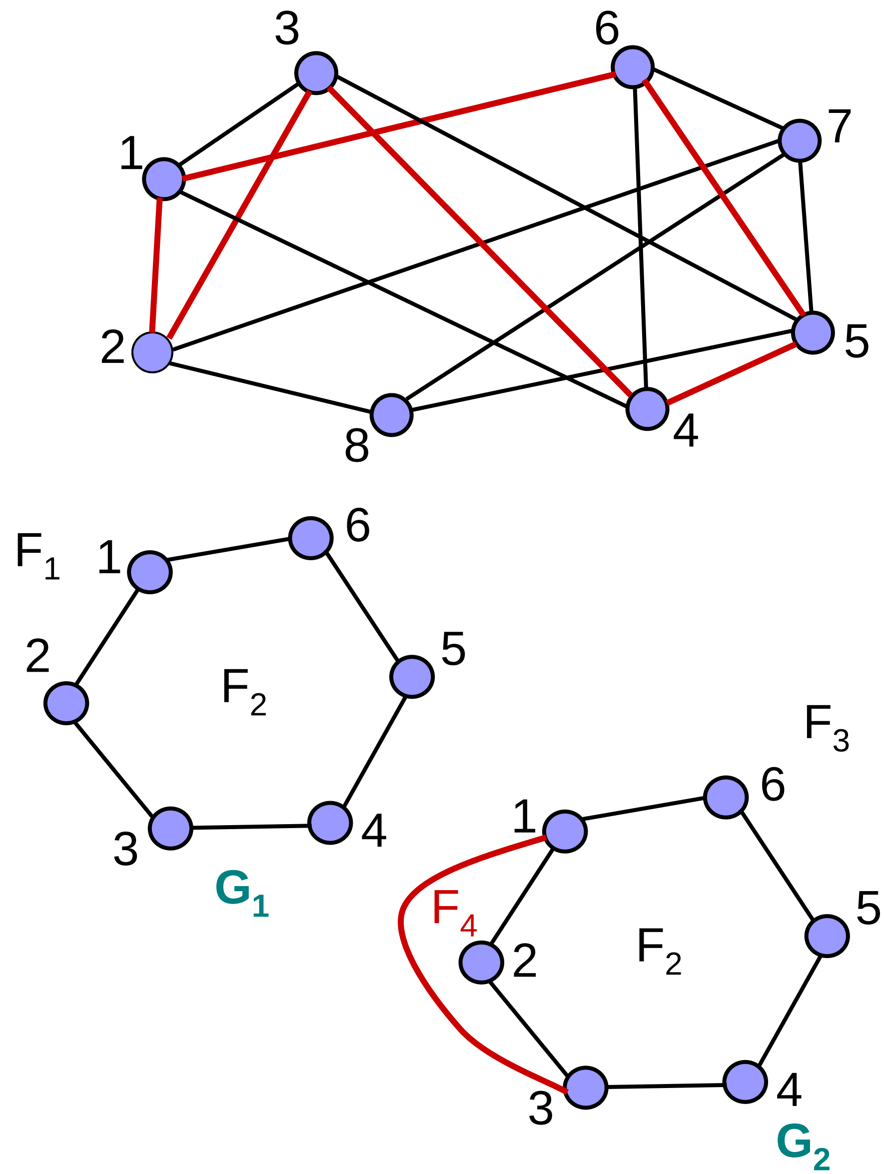
- Let B be any bridge of G relative to H.
- B can be drawn in a face *f* of H', if all the points of contact of B are in the boundary of *f*.
- F(B,H): Set of faces of H' in which B is drawable.
- The algorithm finds a sequence of graphs $G_1, G_2, \ldots,$ such that $G_i \subset G_{i+1}$.
- If G is non-planar then the algorithm stops with the discovery of some bridge B, for which $F(B,G_i) = \emptyset$

Planarity testing algorithm

```
Find a circuit C of G;
i = 1; embeddable = true; G1 = C; f = 2;
while f <> e–n+2 and embeddable do
  find each bridge B of G relative to Gi;
  for each B find F(B,Gi);
  if for some B, F(B,Gi)= ∅ then
    embeddable = false;
    output ‘G is non-planar’;
  endif
  if embeddable then
    if for some B, |F(B,Gi)| = 1 then f = F(B,Gi);
    else
      Let B be any bridge and f be any face, f ∈ F(B,Gi);
    endif
    Find a path Pi⊆B connecting two points of contact of B to Gi;
    G(i+1) = Gi + Pi;
    Draw Pi in the face f of Gi;
    i = i+1; f = f+1;
    if f = e-n+2 then output ‘G is planar’;
  endif
endwhile
```

Example

Gi	f	Bridges	F(B,Gi)	B	F	Pi
G_1	2	B_1	$\{F_1,F_2\}$	B_1	F_1	(1,3)
		B_2	$\{F_1,F_2\}$			
		B_3	$\{F_1,F_2\}$			
		B_4	$\{F_1,F_2\}$			
		B_5	$\{F_1,F_2\}$			
G_2	3	B_2	$\{F_2,F_3\}$	B_5	F_2	(2,7,5)
		B_3	$\{F_2,F_3\}$			
		B_4	$\{F_2,F_3\}$			
		B_5	$\{F_2\}$			



Bridge definitions:

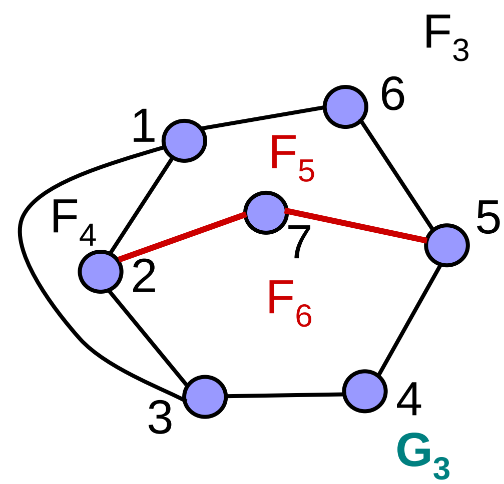
$B_1 = \{(1,3)\}$ $B_2 = \{(1,4)\}$

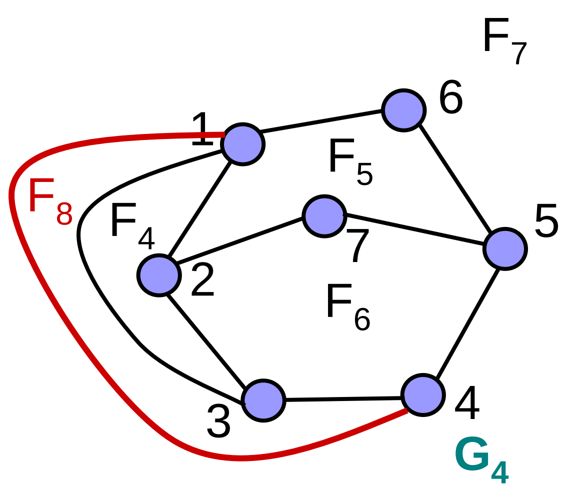
$B_3 = \{(3,5)\}$ $B_4 = \{(4,6)\}$

$B_5 = \{(7,2), (7,5), (7,6), (7,8), (8,2), (8,5)\}$

Example

Gi	f	Bridges	F(B,Gi)	B	F	Pi
G_3	4	B_2	$\{F_3\}$	B_2	F_3	(1,4)
		B_3	$\{F_3,F_6\}$			
		B_4	$\{F_3\}$			
		B_6	$\{F_5\}$			
		B_7	$\{F_5,F_6\}$			
G_4	5	B_3	$\{F_6\}$	B_3	F_6	(3,5)
		B_4	$\{F_7\}$			
		B_6	$\{F_5\}$			
		B_7	$\{F_5,F_6\}$			





Bridge definitions:

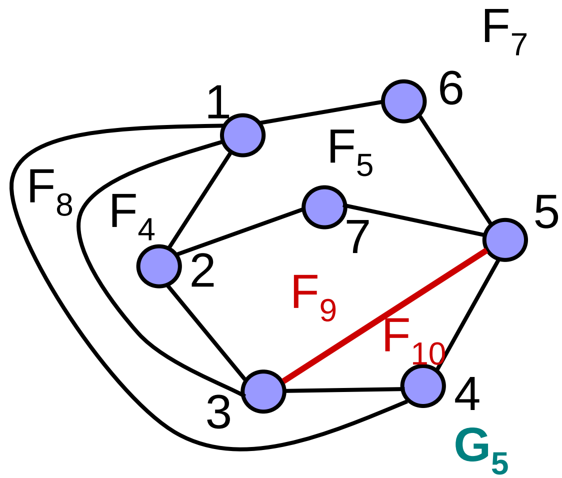
$B_2 = \{(1,4)\}$ $B_3 = \{(3,5)\}$

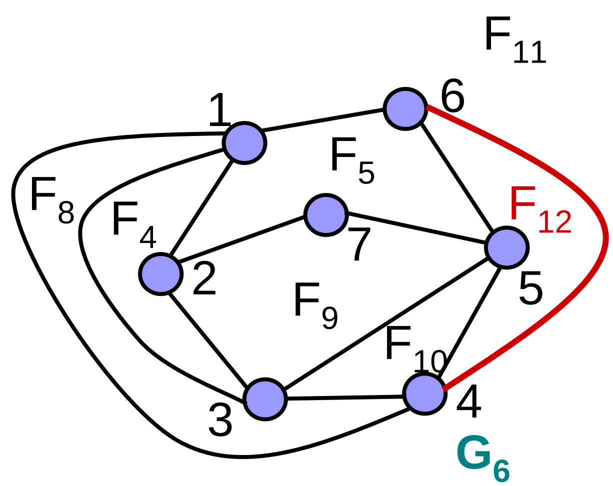
$B_4 = \{(4,6)\}$ $B_6 = \{(6,7)\}$

$B_7 = \{(8,2), (8,5), (8,7)\}$

Example

Gi	f	Bridges	F(B,Gi)	B	F	Pi
G_5	6	B_4 B_6 B_7	$\{F_7\}$ $\{F_5\}$ $\{F_5,F_9\}$	B_4	F_7	(4,6)
G_6	7	B_6 B_7	$\{F_5\}$ $\{F_5,F_9\}$	B_6	F_5	(6,7)





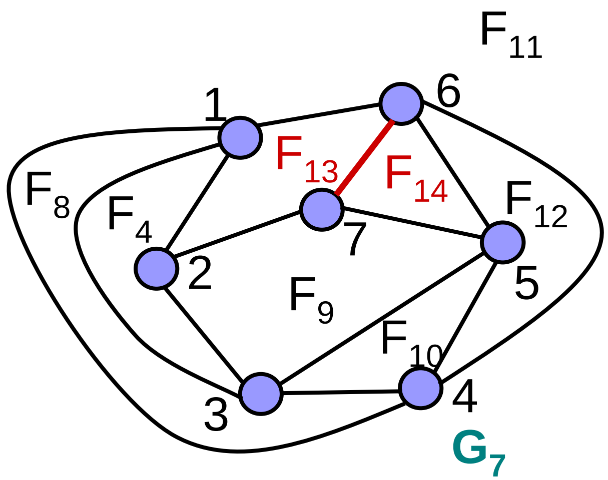
Bridge definitions:

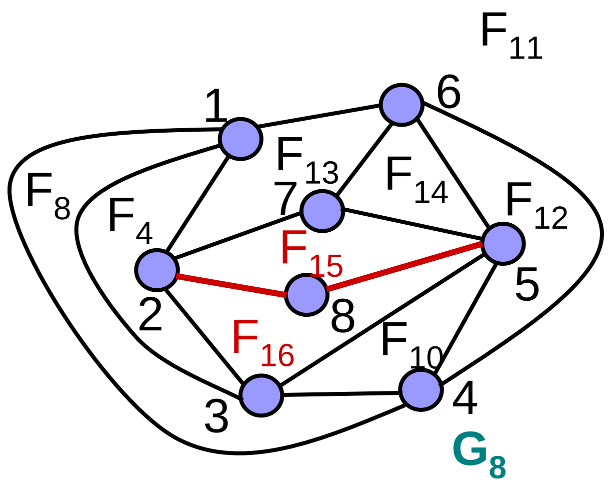
$B_4 = \{(4,6)\}$ $B_6 = \{(6,7)\}$

$B_7 = \{(8,2), (8,5), (8,7)\}$

Example

Gi	f	Bridges	F(B,Gi)	B	F	Pi
G_7	8	B_7	$\{F_9\}$	B_7	F_9	(2,8,5)
G_8	9	B_8	$\{F_{15}\}$	B_8	F_{15}	(7,8)

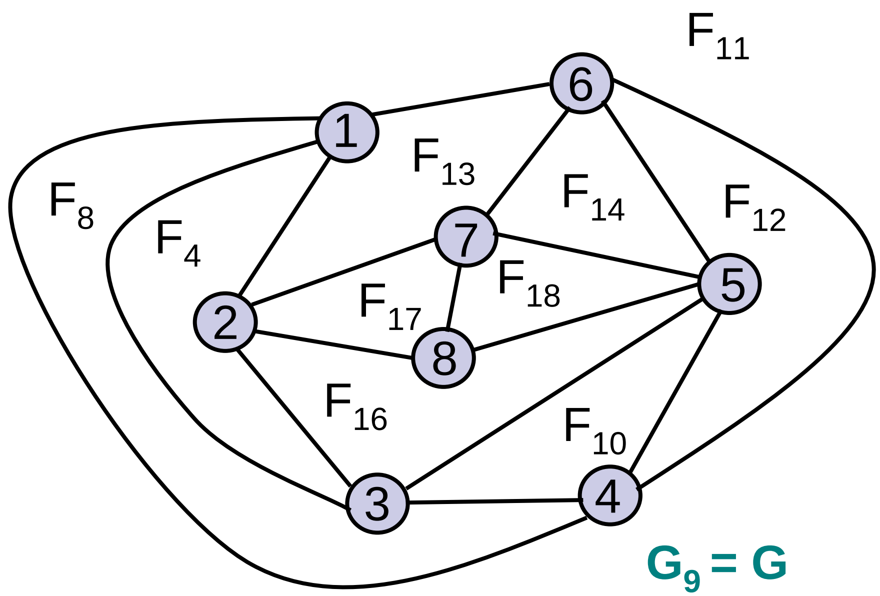




Bridge definitions:
$B_7 = \{(8,2), (8,5), (8,7)\}$
$B_8 = \{(7,8)\}$

Example

- Algorithm terminates when $f = e - n + 2$:
  16 – 8 + 2 = 10 = f



Home study:

- Go to www.planarity.net and play the game!