Chapter 1

Model Theory

1.1. Introduction

Physical simulation of a hydraulic phenomenon, such as the flow over a spillway, in the laboratory is called *physical model* or only *model*. *Prototype* is the hydraulic phenomena in the nature like the spillway over a dam.

Models and prototypes may have geometric, kinematic and dynamic similarities.

Geometric Similarity is the geometric replica of the prototype and it is similarity in form. If L_m= A length on the model, L_p= The length of that physical length on the prototype, the *Length Scale (L_r)* of the model is,

$$L_r = \frac{L_m}{L_p} \qquad (1.1)$$

If the length scale of a prototype and its model is taken as constant, they are geometrically similar. The choose of length scale for a model depends upon to the physical capacity of the laboratory and also to the problem to be examined. Usually 1/20, 1/50, 1/100 scales are used for hydraulic models.

Kinematic Similarity denotes similarity of motion, i.e. similarity of velocity and acceleration components along the x, y, z axes.

Let's take the velocities at conjugant points on the model and the prototype as V_m= Velocity on the model, V_p= Velocity on the prototype, *Velocity Scale* is computed by,

$$V_r = \frac{V_m}{V_p} \qquad (1.2)$$

The model and prototype is *kinematically similar* if the velocity scale is constant. Velocity scale is also can be derived as,

$$V_m = \frac{L_m}{T_m} \quad , \qquad V_p = \frac{L_p}{T_p}$$

$$V_r = \frac{V_m}{V_p} = \frac{L_m}{T_m} \times \frac{T_p}{L_p} = \frac{L_m}{L_p} \times \frac{T_p}{T_m}$$

$$V_r = \frac{L_r}{T_r} \tag{1.3}$$

Where T_r is the *Time Scale***.**

$$T_r = \frac{T_m}{T_p} \tag{1.4}$$

Acceleration Scale can be derived similarly as,

$$a_m = \frac{V_m}{T_m} = \frac{V_r \times V_p}{T_r \times T_p} = \frac{V_r}{T_r} \times a_p = a_r \times a_p$$

$$a_r = \frac{a_m}{a_p} \tag{1.5}$$

$$a_r = \frac{V_r}{T_r} = \frac{L_r}{T_r^2}$$

Acceleration scale will be constant and the paths of fluid particles will be similar when the kinematic similarity is supplied in the model and the prototype.

Dynamic similarity denotes the similarity of forces. If there is a constant ratio between the forces on the conjugant points on the model and the prototype then the two systems is dynamically similar.

$$Force\ Scale = F_r = \frac{F_m}{F_p} = cons \tag{1.6}$$

Generally, inertia, pressure, shearing, gravitational forces are seen on the hydraulic models.

1.2. Similarity Conditions

Model experiments are applied for almost every important hydraulic structure. Optimum solutions may be obtained by the observations and the measurements of the physical event on the model which can not always be seen and understood during the analytical solution of the structure. Hydropower plants, river improvement, coastal engineering and also in aviation and in ship construction sectors are where the model experiments are applied.

Similarity conditions should be supplied to find the measured values between the conjugant points on the model and on the prototype. For instance the measured wave height at a harbor model will correspond which height on the prototype is the question to be answered. *Dynamic Method* is applied to supply the similarity conditions.

1.2.1. Dynamic Method

Dynamic Method depends on the constant variation of forces on the model and on the prototype which is Force Scale is kept constant. Dominant forces on hydraulic structures are *Inertia, Gravitational* and *Viscosity* forces. Denoting as Inertia Force=F_{ine}, Gravitational Force=F_{gr}, Viscosity Force=F_{vis}, the similarity of the ratio of forces can be written as,

$$\frac{(F_{ine})_m}{(F_{ine})_p} = \frac{(F_{gr})_m}{(F_{gr})_p} = \frac{(F_{vis})_m}{(F_{vis})_m} \qquad (1.7)$$

This equality of force ratios equation can also be written as,

$$\begin{aligned} \frac{(F_{gr})_m}{(F_{ine})_m} &= \frac{(F_{gr})_p}{(F_{ine})_p} \\ \frac{(F_{vis})_m}{(F_{ine})_m} &= \frac{(F_{vis})_p}{(F_{ine})_p} \end{aligned} \qquad (1.8)$$

Inertia force can be defined by using Newton's 2nd law as,

Inertia Force = Mass × Acceleration

By using the dimensions of these physical values, the dimension of the inertia force can be derived as,

$$\begin{aligned}
\left[F_{ine}\right] &= [\rho]\times[g]\times\left[L^3\right] \\
\left[F_{ine}\right] &= \left[FL^{-4}T^2\right]\times\left[LT^{-2}\right]\times\left[L^3\right] \\
\left[F_{ine}\right] &= [F]
\end{aligned} \qquad (1.9)$$

Gravitational force is the weight of the body and can be defined as,

Gravitational force = Specific weight × Volume

$$\begin{aligned}
\left[F_{gr}\right] &= \left[FL^{-3}\right]\times\left[L^3\right] \\
\left[F_{gr}\right] &= [F]
\end{aligned} \qquad (1.10)$$

The ratio of forces can be written again by using the above derived equations as,

$$\begin{aligned}
\frac{\left(F_{gr}\right)_m}{\left(F_{ine}\right)_m} &= \frac{\left(F_{gr}\right)_p}{\left(F_{ine}\right)_p} \\
\frac{\rho_m g L_m^3}{\rho_m L_m^3 L_m T_m^{-2}} &= \frac{\rho_p g L_p^3}{\rho_p L_p^3 L_p T_p^{-2}} \qquad (1.11) \\
\frac{gT_m^2}{L_m} &= \frac{gT_p^2}{L_p}
\end{aligned}$$

Since dimension of velocity is,

$$[V] = \frac{[L]}{[T]} \rightarrow [T] = \frac{[L]}{[V]}$$

$$\frac{gL_m^2}{V_m^2 L_m} = \frac{gL_p^2}{V_p^2 L_p}$$

Taking the inverse of this equation gives,

$$\frac{V_m^2}{gL_m} = \frac{V_p^2}{gL_p} \qquad (1.12)$$

Since Froude number (Fr) is,

$$Fr = \frac{V}{\sqrt{gL}}$$

Equation (1.12) shows that Froude numbers calculated for the model and the prototype at a point should be the same.

$$Fr_m = Fr_p \qquad (1.13)$$

The equality of the ratio of the gravity forces to the inertia forces results in to the equality of Froude numbers.

Viscosity force can be defined as,

Viscosity force = Shearing stress × Area

$$\begin{aligned} F_{vis} &= \tau \times A \\ F_{vis} &= \mu \times \frac{du}{dy} \times A \qquad (1.14) \\ [F_{vis}] &= \mu \times \frac{[LT^{-1}]}{[L]} \times [L^2] \end{aligned}$$

The equality of the ratio of viscosity forces to inertia forces is then,

$$\begin{aligned} \frac{(F_{vis})_m}{(F_{ine})_m} &= \frac{(F_{vis})_p}{(F_{ine})_p} \\ \frac{\mu_m \left(L_m T_m^{-1} / L_m\right) L_m^2}{\rho_m L_m^3 \left(L_m T_m^{-2}\right)} &= \frac{\mu_p \left(L_p T_p^{-1} / L_p\right) L_p^2}{\rho_p L_p^3 \left(L_p T_p^{-2}\right)} \qquad (1.15) \\ \frac{\mu_m T_m}{\rho_m L_m^2} &= \frac{\mu_p T_p}{\rho_p L_p^2} \end{aligned}$$

Since,

$$[V]=\frac{[L]}{[T]}\rightarrow[T]=\frac{[L]}{[V]}$$

Equation can be written as,

$$\frac{\mu_m L_m}{\rho_m L_m^2 V_m}=\frac{\mu_p L_p}{\rho_p L_p^2 V_p}$$

Taking the inverse of this equation yields,

$$\frac{\rho_m V_m L_m}{\mu_m}=\frac{\rho_p V_p L_p}{\mu_p} \qquad (1.16)$$

Since Reynolds number (Re) is,

$$\mathrm{Re}=\frac{\rho VL}{\mu}$$

Equation (1.16) shows the equality of Reynolds numbers for the model and the prototype at the point taken.

$$\mathrm{Re}_m=\mathrm{Re}_p \qquad (1.17)$$

The derived results are,

1. Froude and Reynolds numbers are the ratio of,

Froude number = Fr = Inertial force / Gravitational force

Reynolds number = Re = Inertial force / Viscosity force

2. Dynamic similarity can be supplied by the equality of Froude and Reynolds numbers simultaneously at the model and the prototype.

$$Fr_m = Fr_p \quad , \quad \mathrm{Re}_m = \mathrm{Re}_p$$

1.3. Selection of Model Scale

Using the equality of Froude numbers,

$$\begin{aligned}
&Fr_m = Fr_p \rightarrow Fr_m^2 = Fr_p^2 \\
&\frac{V_m^2}{g_m L_m} = \frac{V_p^2}{g_p L_p} \\
&\frac{V_m^2}{V_p^2} \times \frac{g_p}{g_m} \times \frac{L_p}{L_m} = 1 \\
&\frac{V_r^2}{g_r L_r} = 1
\end{aligned} \qquad (1.18)$$

Sine the model and the prototype will be constructed on the same planet (earth), *gravitational acceleration scale* is $g_r = 1$. The above equation gives the mathematical relation between the velocity and geometric scale as,

$$V_r^2 = L_r \qquad (1.19)$$

This mathematical is the result of the equality of Froude numbers at the model and the prototype.

Using the equality of Reynolds numbers,

$$\mathrm{Re}_m = \mathrm{Re}_p$$

$$\frac{\rho_m V_m L_m}{\mu_m} = \frac{\rho_p V_p L_p}{\mu_p}$$

$$\nu = \frac{\mu}{\rho}$$

$$\frac{V_m L_m}{\nu_m} = \frac{V_p L_p}{\nu_p} \qquad (1.20)$$

$$\frac{V_m}{V_p} \times \frac{L_m}{L_p} \times \frac{\nu_p}{\nu_m} = 1$$

$$\frac{V_r L_r}{\nu_r} = 1$$

Since the same fluid (water) will be used at the model and the prototype, *kinematic viscosity scale* is $\upsilon_r = 1$. The equality of Reynolds numbers yields the mathematical relation between velocity and geometric scale as,

$$V_r L_r = 1 \qquad (1.21)$$

Since the equality of Reynolds and Froude numbers must be supplied simultaneously, using the derived relations between the velocity and geometric scales,

$$\begin{aligned} V_r^2 &= L_r \\ V_r L_r &= 1 \\ L_r &= 1 \\ L_m &= L_p \end{aligned} \qquad (1.22)$$

This result shows that the model and the prototype will be at the same size which does not have any practical meaning at all. The ratio of gravitational, inertial and viscosity forces **can not** be supplied at the same time. One of the viscosity or gravitational forces is taken into consideration for the model applications.

1.4. Froude Models

Froude models denote supplying the equality of Froude numbers at the model and the prototype. Open channel models are constructed as Froude model since the motive force in open channels is gravity force which is the weight of water in the flow direction. Dam spillways, harbors, water intake structures and energy dissipators are the examples of hydraulic structures.

Taking the derived relation between the velocity and geometric scales,

$$V_r^2 = L_r \rightarrow V_r = L_r^{1/2} \qquad (1.23)$$

The scales of the other physical variables can be derived in geometric scale. The dimension of a physical variable A,

$$[A] = [L^x] \times [M^y] \times [T^z]$$

The dimension of this physical value A is then,

$$A_r = L_r^x M_r^y T_r^z \qquad (1.24)$$

Since,

$$\begin{aligned} M_r &= \rho_r L_r^3 \\ \rho_r &= 1 \\ M_r &= L_r^3 \end{aligned} \qquad (1.25)$$

$$T_r = \frac{L_r}{V_r} = \frac{L_r}{L_r^{1/2}} = L_r^{1/2} \qquad (1.26)$$

The scale of the physical value of A is,

$$\begin{aligned} A_r &= L_r^x L_r^{3y} L_r^{z/2} \\ A_r &= L_r^{x+3y+z/2} \end{aligned} \qquad (1.27)$$

The discharge scale can be found as by using the above derived equation as,

$$[Q] = [L^3][M^0][T^{-1}]$$

$$Q_r = L_r^{3-1/2} \qquad (1.27)$$

$$Q_r = L_r^{5/2}$$

Example 1.1: Geometric scale of a spillway model is chosen as $L_r = 1/10$.

a) If the discharge on the prototype is $Q_p = 100\ m^3/sec$, what will be the discharge on the model?

Since,

$$Q_r = L_r^{5/2} = \left(\frac{1}{10}\right)^{5/2}$$

$$Q_r = \frac{1}{316.2}$$

$$Q_r = \frac{Q_m}{Q_p}$$

$$Q_m = \frac{100}{316.2}$$

$$Q_m = 0.316 m^3 / \sec$$

b) If the velocity at a point on the model is measured as $V_m = 3$ m/sec, what will be the velocity on the prototype at that homolog point?

$$V_r = L_r^{1/2}$$

$$V_r = \left(\frac{1}{10}\right)^{1/2} = \frac{1}{3.16}$$

$$V_r = \frac{V_m}{V_p}$$

$$V_p = \frac{V_m}{V_r} = 3 \times 3.16 = 9.49 m / \sec$$

c) If the energy dissipated with hydraulic jump on the basin of the spillway is N = 100 kW, what will be the dissipated energy on the prototype?

Energy equation is,

$$
\begin{aligned}
&N = \gamma QH \\
&[N] = [ML^{-2}T^{-2}][L^3T^{-1}][L] \\
&[N] = [L^2MT^{-3}]
\end{aligned}
$$

Using the derived scale equation,

$$A_r = L_r^x M_r^y T_r^z = L_r^{x+3y+z/2}$$

$$N_r = L_r^{2+3-3/2} = L_r^{7/2}$$

Power scale is,

$$N_r = \left(\frac{1}{10}\right)^{7/2} = \frac{1}{3162}$$

The dissipated energy on the basin is,

$$
\begin{aligned}
&N_r = \frac{N_m}{N_p} \\
&N_p = 100 \times 3162 = 316200 kW
\end{aligned}
$$

Example 1.2: Prototype discharge has been given as Q = 3 m^3/sec for a Froude model. Velocity and force at a point on the model have been measured as V_m = 0.2 m/sec and F_m = 1 N. Calculate the discharge for the model and velocity and force at the conjugate point on the prototype. Geometric scale has been as L_r = 1/100. The same liquid will be used at both model and prototype.

Solution:

$$L_r = \frac{L_m}{L_p} = \frac{1}{100}$$

Discharge for the model is,

$$Q_r = L_r^{5/2}$$

$$Q_r = \left(\frac{1}{100}\right)^{2.5} = 10^{-5} = \frac{Q_m}{Q_p}$$

$$Q_m = 10^{-5} \times 3 = 3 \times 10^{-5}\ m^3/\sec = 0.03\, lt/\sec$$

Velocity scale is,

$$V_r = L_r^{1/2} = \left(\frac{1}{100}\right)^{0.5} = 10^{-1}$$

$$\frac{V_m}{V_p} = 10^{-1}$$

$$V_p = \frac{0.2}{10^{-1}} = 2\, m/\sec$$

Force scale is,

$$F = Ma$$

$$F_r = \rho_r L_r^3 \frac{L_r}{T_r^2}$$

$$V_r = \frac{L_r}{T_r} = L_r^{1/2}$$

$$T_r = L_r^{1/2}$$

$$F_r = \rho_r L_r^3 \frac{L_r}{L_r}$$

$$\rho_r = 1$$

$$F_r = L_r^3$$

$$F_r = \left(\frac{1}{100}\right)^3 = 10^{-6}$$

$$F_r = \frac{F_m}{F_p}$$

$$F_p = \frac{F_m}{F_r} = \frac{1}{10^{-6}} = 10^6\, N = 10^3\, kN$$

1.5. Reynolds Models

If the governing forces of the motion are the viscosity and the inertia forces like in pressured pipe flows, Reynolds models are used. In Reynolds models the equality of the Reynolds numbers are supplied.

Using the derived relation between the velocity and the geometric scale, the scale of any physical value can be derived.

$$\begin{aligned} V_r L_r &= 1 \\ A_r &= L_r^x M_r^y T_r^z \end{aligned} \qquad (1.28)$$

Since,

$$\begin{aligned} M_r &= L_r^3 \\ T_r &= \frac{L_r}{T_r} = \frac{L_r}{L_r^{-1}} = L_r^2 \\ A_r &= L_r^x L_r^{3y} L_r^{2z} \\ A_r &= L_r^{x+3y+2z} \end{aligned} \qquad (1.29)$$

The discharge scale for Reynolds models is,

$$\begin{aligned} [Q] &= [L^3][M^0][T^{-1}] \\ Q_r &= L_r^{3-2} \\ Q_r &= L_r \end{aligned} \qquad (1.30)$$

Discharge is equal to the geometric scale for Reynolds models.

Example 1.3: Q = 0.05 m^3/sec, D_1 = 0.20 m, and D_2 = 0.15 m are in a venturimeter. A model of it will made in L_r = 1/5 geometric scale. Find out time, velocity, discharge, and pressure scale of this Venturimeter.

Figure

Solution: Reynolds model will be applied since the flow is a pressured flow in a Venturimeter.

Geometric has been given as,

$$L_r = \frac{L_m}{L_p} = \frac{1}{5}$$

Time scale is;

$$T_r = \frac{T_m}{T_p}$$
$$T_r = L_r^{x+3y+2z}$$

$$\begin{aligned} &[T] = \left[L^0 M^0 T^1\right] \\ &x = 0, y = 0, z = 1 \qquad (1.31) \\ &T_r = L_r^2 = \frac{1}{25} \end{aligned}$$

Velocity scale is;

$$V_r = \frac{V_m}{V_p}$$
$$V_r = L_r^{x+3y+2z}$$

$$\begin{aligned} &[V] = \left[L^1 M^0 T^{-1}\right] \\ &x = 1, y = 0, z = -1 \qquad (1.32) \\ &V_r = L_r^{1-2} = L_r^{-1} = \frac{1}{L_r} = 5 \end{aligned}$$

Pressure scale is;

$$p_r = \frac{p_m}{p_p}$$
$$[p_r] = \left[L_r^{x+3y+2z}\right]$$

$$[p]=\left[\frac{F}{L^2}\right]$$
$$[F]=\left[M\,\frac{L}{T^2}\right]$$
$$[p]=\left[MLT^{-2}L^{-2}\right]=\left[L^{-1}M^{1}T^{-2}\right]$$

$$x=-1,\ y=1,\ z=-2$$
$$[p_r]=\left[L_r^{-1+3-4}\right]=\left[L_r^{-2}\right]=\left[\frac{1}{L_r^2}\right]=25 \qquad (1.33)$$

Discharge scale is,

$$Q_r=\frac{Q_m}{Q_p}$$
$$[Q_r]=\left[L_r^{x=3y+2z}\right]$$

$$[Q]=\left[\frac{L^3}{T}\right]=\left[L^3M^0T^{-1}\right]$$
$$x=3,\ y=0,\ z=-1$$
$$[Q_r]=\left[L_r^{3+0-2}\right]=[L_r]=\frac{1}{5}$$