Circuit and System Analysis
EEF 232E

Prof. Dr. Müştak E. Yalçın

Istanbul Technical University
Faculty of Electrical and Electronic Engineering

mustak.yalcin@itu.edu.tr

Outline I

1. Circuit Analysis in the s-domain
   - Circuit elements in the s-domain
   - Mesh currents method
   - Node-voltage Method
   - Network Parameters
   - Combinations of two-port networks
   - Reciprocal Network
   - Thevenin - Norton Equivalent Circuits

s-domain equivalent circuit for each circuit element

A resistor in the s-domain

In time-domain

$$v = Ri$$

s-domain equivalent circuit for the resistor

$$V(s) = RI(s)$$

where $V(s) = \mathcal{L}\{v\}$ and $I(s) = \mathcal{L}\{i\}$.

A Capacitor in the s-domain:

Terminal current in time-domain

$$i = C\frac{dv}{dt}$$

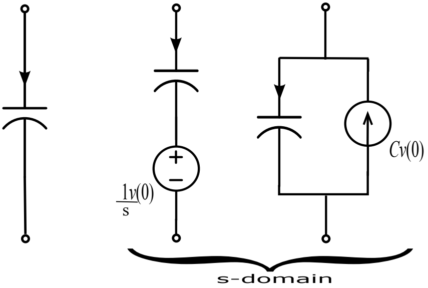
Laplace transform the above equ.

$$I(s) = CsV(s) - Cv(0)$$

or

$$V(s) = \frac{1}{Cs} I + \frac{1}{s} v(0)$$

Norton and Thevenin equivalents of a capacitor in the s-domain



An inductor in the s-domain:

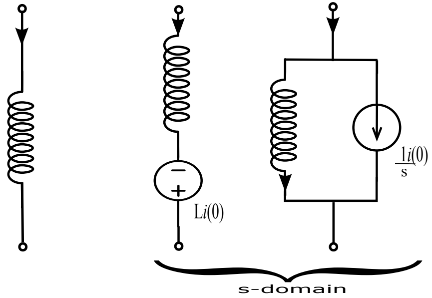
In time-domain

$$L\frac{di}{dt} = v$$

After Laplace transform

$$LsI(s) - Li(0) = V \quad \text{or} \quad I(s) = \frac{1}{Ls}V + \frac{1}{s}i(0)$$

Equivalent circuits for the inductor



Energy is stored in the inductor and capacitor

The s-domain impedance and admintance

If no energy is stored in the inductor or capacitor, the relationship between the terminal voltage and current for each passive element takes the form:

$$V(s) = Z(s)I(s)$$

or

$$I(s) = Y(s)V(s)$$

$Z(s)$ is impedance and $Y(s)$ is admintance function.

Resistor has an impedance of RΩ, an inductor has an impedance of $Ls\Omega$, and a capacitor has an impedance of $1/sC\Omega$

KCL and KVL in s-domain

The algebraic sum of the currents/voltages at the node/loop is zero in time domain, the algebraic sum of the transformed currents/voltages is also zero.

Mesh currents method

Write mesh equations

$$BV(s) = 0$$

Laplace transform of the equ.

$$BV(s) = B_1 V_e(s) + B_2 V_k(s) = 0$$

where $V_k(s)$ and $V_e(s) = [V_R(s) \;\; V_C(s) \;\; V_L(s)]^T$ voltages of independent voltage sources and the others, respectively. The terminal equ.s

$$V_e(s) = \mathbf{Z(s)} I_e(s) + \begin{bmatrix} 0_{n_R \times n_R} & 0 & 0 \\ 0 & \frac{1}{s} I_{n_C \times n_C} & 0 \\ 0 & 0 & -\mathbf{L}_{n_L \times n_L} \end{bmatrix} \begin{bmatrix} i_R \\ v_C(0) \\ i_L(0) \end{bmatrix}$$

Substituting the above equ. into the mesh equ.

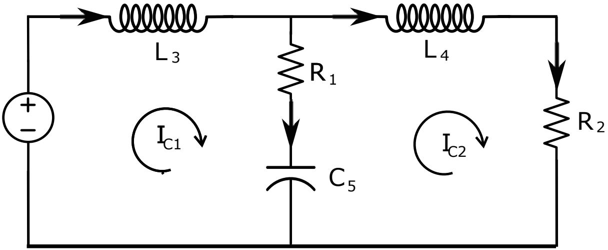
$$B_1 \mathbf{Z} I_e + B_1 \begin{bmatrix} 0 & 0 & 0 \\ 0 & \frac{1}{s}I & 0 \\ 0 & 0 & -LI \end{bmatrix} \begin{bmatrix} i_R \\ v_C(0) \\ i_L(0) \end{bmatrix} + B_2 V_k = 0$$

Hence the mesh currents is obtained

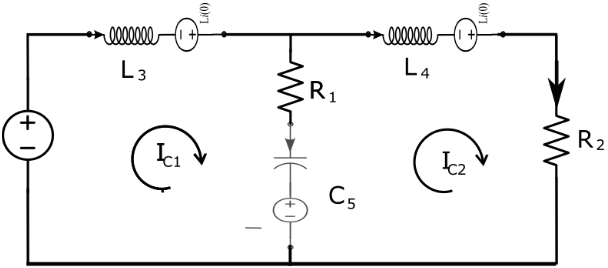
$$B_1 \mathbf{Z} B_1^T I_c(s) + B_1 \begin{bmatrix} 0 & 0 & 0 \\ 0 & \frac{1}{s}I & 0 \\ 0 & 0 & -LI \end{bmatrix} \begin{bmatrix} i_R \\ v_C(0) \\ i_L(0) \end{bmatrix} + B_2 V_k = 0$$

See :EHB211 E

Example



Example



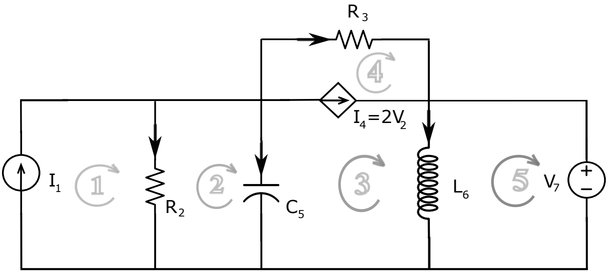
$$M1 \quad L_3 s I_{c1} - L_3 i_{L3}(0) + (\tfrac{1}{Cs} + R)(I_{c1} - I_{c2}) + \tfrac{1}{s} v_C(0) - V_G = 0$$
$$M2 \quad L_4 s I_{c2} - L_4 i_{L4}(0) + R_2 I_{c2} + (\tfrac{1}{Cs} + R)(I_{c2} - I_{c1}) - \tfrac{1}{s} v_C(0) = 0$$

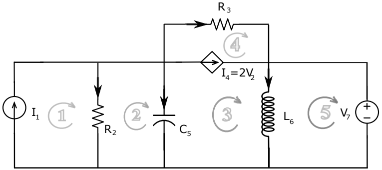
In matrix form

$$\underbrace{\begin{bmatrix} L_3 s & -\frac{1}{Cs} - R_1 \\ -\frac{1}{Cs} - R_1 & L_4 s + R_2 \end{bmatrix}}_{\mathbf{A}} \begin{bmatrix} I_{c1} \\ I_{c2} \end{bmatrix} = \begin{bmatrix} V_G \\ 0 \end{bmatrix} + \begin{bmatrix} -\frac{1}{s} & L_3 & 0 \\ \frac{1}{s} & 0 & L_4 \end{bmatrix} \begin{bmatrix} v_C(0) \\ i_{L3}(0) \\ i_{L4}(0) \end{bmatrix}$$

$$\begin{bmatrix} I_{c1} \\ I_{c2} \end{bmatrix} = \underbrace{A^{-1} \begin{bmatrix} V_G \\ 0 \end{bmatrix}}_{\text{zero-state response}} + \underbrace{A^{-1} \begin{bmatrix} -\frac{1}{s} & L_3 & 0 \\ \frac{1}{s} & 0 & L_4 \end{bmatrix} \begin{bmatrix} v_C(0) \\ i_{L3}(0) \\ i_{L4}(0) \end{bmatrix}}_{\text{zero-input response}}$$

Example





$$\begin{bmatrix} R_2 & -R_2 & 0 & 0 & 0 \\ -R_2 & R_2+\frac{1}{C_5 s} & -\frac{1}{C_5 s} & 0 & 0 \\ 0 & -\frac{1}{C_5 s} & \frac{1}{C_5 s}+L_6 s & 0 & -L_6 s \\ 0 & 0 & 0 & R_3 & 0 \\ 0 & 0 & 0 & -L_6 s & L_6 s \end{bmatrix} \begin{bmatrix} I_{c1} \\ I_{c2} \\ I_{c3} \\ I_{c4} \\ I_{c5} \end{bmatrix}$$

$$= \begin{bmatrix} -V_1 \\ 0 \\ -V_4 \\ V_4 \\ -V_7 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ -1/s & 0 \\ 1/s & L_6 \\ 0 & 0 \\ 0 & L_6 \end{bmatrix} \begin{bmatrix} v_C(0) \\ i_L(0) \end{bmatrix}$$

$$I_1 = I_{c1}$$
$$I_4 = 2V_2$$
$$I_{c3} - I_{c4} = -2V_1$$

$$\begin{bmatrix} R_2 & -R_2 & 0 & 0 & 0 & 1 & 0 \\ -R_2 & R_2 + \frac{1}{C_5 s} & -\frac{1}{C_5 s} & 0 & 0 & 0 & 0 \\ 0 & -\frac{1}{C_5 s} & \frac{1}{C_5 s} + L_6 s & 0 & -L_6 s & 0 & 1 \\ 0 & 0 & 0 & R_3 & 0 & 0 & -1 \\ 0 & 0 & -L_6 s & 0 & L_6 s & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 2 & 0 \end{bmatrix} \begin{bmatrix} I_{c1} \\ I_{c2} \\ I_{c3} \\ I_{c4} \\ I_{c5} \\ V_1 \\ V_4 \end{bmatrix}$$
$$= \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ -V_7 \\ I_1 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ -1/s & 0 \\ 1/s & L_6 \\ 0 & 0 \\ 0 & L_6 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_C(0) \\ i_L(0) \end{bmatrix}$$

Write the fundamental cut-set equations for the nodes which do not correspond to node of a voltage sources:

$$AI(s) = 0$$

$$A_1 I_e(s) + A_2 I_k(s) = 0$$

where $I_k(s)$ currents of current sources and the currents of others $I_e = [I_R \;\; I_C \;\; I_L]$

$$I_e = \mathbf{Y(s)} V_e + \begin{bmatrix} 0 & 0 & 0 \\ 0 & -\mathbf{C} & 0 \\ 0 & 0 & \frac{1}{s} I \end{bmatrix} \begin{bmatrix} V_R \\ v_C(0) \\ i_L(0) \end{bmatrix}$$

Substituting

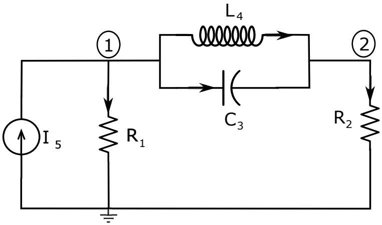
$$A_1 \mathbf{Y(s)} V_e + A_1 \begin{bmatrix} 0 & 0 & 0 \\ 0 & -\mathbf{C} & 0 \\ 0 & 0 & \frac{1}{s} I \end{bmatrix} \begin{bmatrix} V_R \\ v_C(0) \\ i_L(0) \end{bmatrix} + A_2 i_k = 0$$

Using $V_e = A_1^T V_d$, we obtain the node voltage

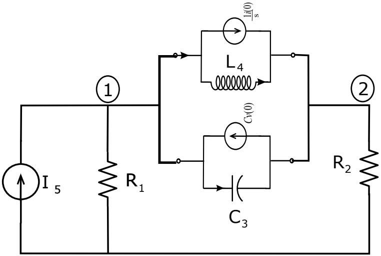
$$A_1 \mathbf{Y}(\mathbf{s}) A_1^T V_d + A_1 \begin{bmatrix} 0 & 0 & 0 \\ 0 & -\mathbf{C} & 0 \\ 0 & 0 & \frac{1}{s} I \end{bmatrix} \begin{bmatrix} V_R \\ v_C(0) \\ i_L(0) \end{bmatrix} + A_2 i_k = 0$$

See :EHB211 E

Example

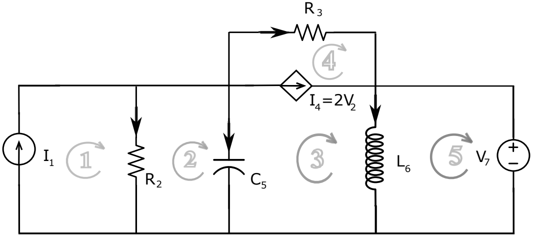


Example



$$\begin{bmatrix} G_1 + C_3 s + \frac{1}{L_3 s} & -C_3 s - \frac{1}{L_3 s} \\ -C_3 s - \frac{1}{L_3 s} & G_2 + C_3 s + \frac{1}{L_3 s} \end{bmatrix} \begin{bmatrix} V_{d1} \\ V_{d2} \end{bmatrix} = \begin{bmatrix} I_5 \\ 0 \end{bmatrix} + \begin{bmatrix} C_3 & -\frac{1}{s} \\ -C_3 & \frac{1}{s} \end{bmatrix} \begin{bmatrix} v_3(0) \\ i_4(0) \end{bmatrix}$$

Example

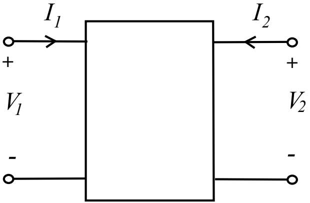


$$\begin{bmatrix} G_2+G_3+C_5s & -G_3 \\ -G_3 & G_3+\frac{1}{L_6s} \end{bmatrix} \begin{bmatrix} V_{d1} \\ V_{d2} \end{bmatrix} = \begin{bmatrix} I_1 - I_4 \\ I_4 - I_7 \end{bmatrix} + \begin{bmatrix} 1/s & 0 \\ 0 & -L_6 \end{bmatrix} \begin{bmatrix} v_5(0) \\ i_6(0) \end{bmatrix}$$

$$V_7 = V_{d1}$$
$$I_4 = 2V_2 = 2V_{d1}$$

$$\begin{bmatrix} V_{d1} \\ V_{d2} \\ I_7 \end{bmatrix} = \underbrace{\begin{bmatrix} G_2 + 2 + G_3 + C_5 s & -G_3 & 0 \\ -G_3 - 2 & G_3 + \frac{1}{L_6 s} & 1 \\ 1 & 0 & 0 \end{bmatrix}^{-1} \begin{bmatrix} I_1 \\ 0 \\ V_7 \end{bmatrix}}_{\text{zero-state response}}$$

$$+ \underbrace{\begin{bmatrix} G_2 + 2 + G_3 + C_5 s & -G_3 & 0 \\ -G_3 - 2 & G_3 + \frac{1}{L_6 s} & 1 \\ 1 & 0 & 0 \end{bmatrix}^{-1} \begin{bmatrix} 1/s & 0 \\ 0 & -L_6 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_5(0) \\ i_6(0) \end{bmatrix}}_{\text{zero-input response}}$$



Impedance matrix (z-parameter) The two currents I_1 and I_2 are assumed to be known, and the voltages V_1 and V_2 can be found by:

$$\begin{bmatrix} V_1(s) \\ V_2(s) \end{bmatrix} = \begin{bmatrix} z_{11} & z_{12} \\ z_{21} & z_{22} \end{bmatrix} \begin{bmatrix} I_1(s) \\ I_2(s) \end{bmatrix}$$

where

$$z_{11} = \left.\frac{V_1(s)}{I_1(s)}\right|_{I_2=0} \qquad z_{12} = \left.\frac{V_1(s)}{I_2(s)}\right|_{I_1=0}$$

$$z_{21} = \left.\frac{V_2(s)}{I_1(s)}\right|_{I_2=0} \qquad z_{22} = \left.\frac{V_2(s)}{I_2(s)}\right|_{I_1=0}$$

Network Parameters

Admittance matrix (y-parameters) The two voltages V_1 and V_2 are assumed to be known, and the currents I_1 and I_2 can be found by:

$$\begin{bmatrix} I_1(s) \\ I_2(s) \end{bmatrix} = \begin{bmatrix} y_{11} & y_{12} \\ y_{21} & y_{22} \end{bmatrix} \begin{bmatrix} V_1(s) \\ V_2(s) \end{bmatrix}$$

where

$$y_{11} = \left.\frac{I_1(s)}{V_1(s)}\right|_{V_2=0} \qquad y_{12} = \left.\frac{I_1(s)}{V_2(s)}\right|_{V_1=0}$$

$$y_{21} = \left.\frac{I_2(s)}{V_1(s)}\right|_{V_2=0} \qquad y_{22} = \left.\frac{I_2(s)}{V_2(s)}\right|_{V_1=0}$$

y_{21} and y_{12} are transfer admittances

g-parameters

Inverse hybrid model, we assume V_1 and I_2 are known, and find V_2 and I_1 by :

$$\begin{bmatrix} I_1(s) \\ V_2(s) \end{bmatrix} = \begin{bmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{bmatrix} \begin{bmatrix} V_1(s) \\ I_2(s) \end{bmatrix}$$

where

$$g_{11} = \left.\frac{I_1(s)}{V_1(s)}\right|_{I_2=0} \qquad g_{12} = \left.\frac{I_1(s)}{I_2(s)}\right|_{V_1=0}$$

$$g_{21} = \left.\frac{V_2(s)}{V_1(s)}\right|_{I_2=0} \qquad g_{22} = \left.\frac{V_2(s)}{I_2(s)}\right|_{V_1=0}$$

h-parameters

Hybrid model, we assume V_2 and I_1 are known, and find V_1 and I_2 by:

$$\begin{bmatrix} V_1(s) \\ I_2(s) \end{bmatrix} = \begin{bmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{bmatrix} \begin{bmatrix} I_1(s) \\ V_2(s) \end{bmatrix}$$

where

$$h_{11} = \left.\frac{V_1(s)}{I_1(s)}\right|_{V_2=0} \qquad h_{12} = \left.\frac{V_1(s)}{V_2(s)}\right|_{I_1=0}$$

$$h_{21} = \left.\frac{I_2(s)}{I_1(s)}\right|_{V_2=0} \qquad h_{22} = \left.\frac{I_2(s)}{V_2(s)}\right|_{I_1=0}$$

ABCD-parameters

Transmission model, we assume V_1 and I_1 are known, and find V_2 and I_2 by:

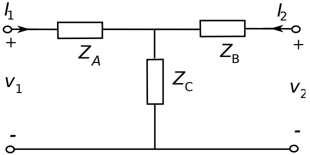
$$\begin{bmatrix} V_1(s) \\ I_1(s) \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} V_2(s) \\ -I_2(s) \end{bmatrix}$$

where

$$A = \left.\frac{V_1(s)}{V_2(s)}\right|_{I_2=0} \qquad B = \left.\frac{V_1(s)}{-I_2(s)}\right|_{V_2=0}$$

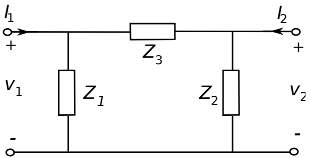
$$C = \left.\frac{I_1(s)}{V_2(s)}\right|_{I_2=0} \qquad D = \left.\frac{-I_1(s)}{I_2(s)}\right|_{V_2=0}$$

Example



$$z_{11} = \left.\frac{V_1(s)}{I_1(s)}\right|_{I_2=0} = Z_A + Z_C \qquad z_{12} = \left.\frac{V_1(s)}{I_2(s)}\right|_{I_1=0} = Z_C$$

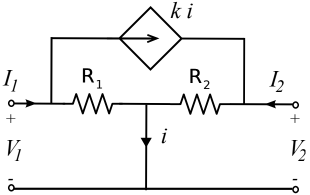
$$z_{21} = \left.\frac{V_2(s)}{I_1(s)}\right|_{I_2=0} = Z_C \qquad z_{22} = \left.\frac{V_2(s)}{I_2(s)}\right|_{I_1=0} = Z_B + Z_C$$



ABCD

$$1+\frac{Z_3}{Z_2} \quad Z_3 \frac{1}{Z_1}+\frac{1}{Z_2}+\frac{Z_3}{Z_1 Z_2} \quad 1+\frac{Z_3}{Z_1}$$

Example

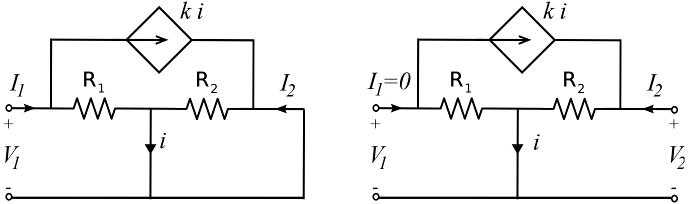


For h-parameters

$$h_{11} = \left.\frac{V_1(s)}{I_1(s)}\right|_{V_2=0} \quad \text{Port 2 short circuit}$$

$$h_{21} = \left.\frac{I_2(s)}{I_1(s)}\right|_{V_2=0} \quad \text{Port 2 short circuit}$$

$$h_{11} = \left.\frac{V_1(s)}{I_1(s)}\right|_{V_2=0} \qquad h_{21} = \left.\frac{I_2(s)}{I_1(s)}\right|_{V_2=0}$$



Input current ($i = I_1$)

$$I_1 = k\frac{V_1}{R_1} + \frac{V_1}{R_1}$$

then

$$h_{11} = \frac{R_1}{1+k}$$

for h_{21}

$$I_2 = -ki$$

Substituing $I_1 = (k+1)i$ into the above equ

$$h_{21} = -\frac{k}{1+k}$$

Problems: Richard C. Dorf, James A. Svoboda-Introduction to Electric Circuits-Wiley (2013) page: 859

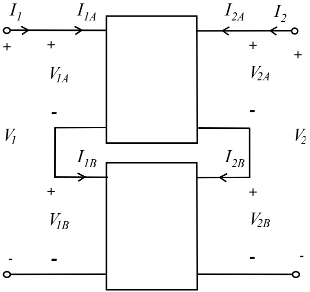
The rest of the parameters

$$h_{12} = \left.\frac{V_1(s)}{V_2(s)}\right|_{I_1=0} \qquad h_{22} = \left.\frac{I_2(s)}{V_2(s)}\right|_{I_1=0}$$

For $I_1 = 0$, the circuit is given

$$h_{12} = -\frac{R_1 k}{(1+k)R_2} \qquad h_{22} = \frac{1}{(1+k)R_2}$$

Series-series connection



From KCL and KVL

$$I_1 = I_{1A} = I_{1B} \qquad I_2 = I_{2A} = I_{2B}$$

and

$$V_1 = V_{1A} + V_{1B} \qquad V_2 = V_{2A} + V_{2B}$$

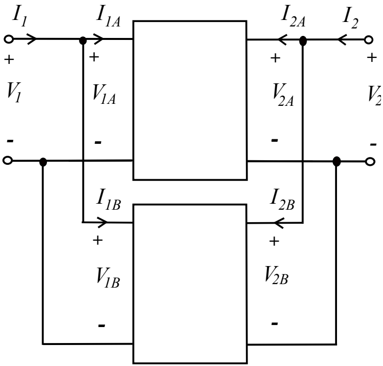
$$\begin{bmatrix} V_1(s) \\ V_2(s) \end{bmatrix} = \begin{bmatrix} V_{1A}(s) \\ V_{2A}(s) \end{bmatrix} + \begin{bmatrix} V_{1B}(s) \\ V_{2B}(s) \end{bmatrix} = \begin{bmatrix} z_{11A} & z_{12A} \\ z_{21A} & z_{22A} \end{bmatrix} \begin{bmatrix} I_{1A}(s) \\ I_{2A}(s) \end{bmatrix}$$

$$+ \begin{bmatrix} z_{11B} & z_{12B} \\ z_{21B} & z_{22B} \end{bmatrix} \begin{bmatrix} I_{1B}(s) \\ I_{2B}(s) \end{bmatrix}$$

$$= \left\{ \begin{bmatrix} z_{11A} & z_{12A} \\ z_{21A} & z_{22A} \end{bmatrix} + \begin{bmatrix} z_{11B} & z_{12B} \\ z_{21B} & z_{22B} \end{bmatrix} \right\} \begin{bmatrix} I_1(s) \\ I_2(s) \end{bmatrix}$$

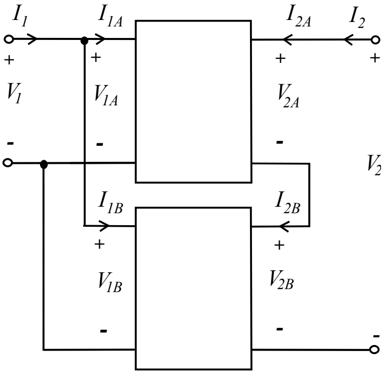
$$Z = Z_A + Z_B$$

Parallel-parallel connection



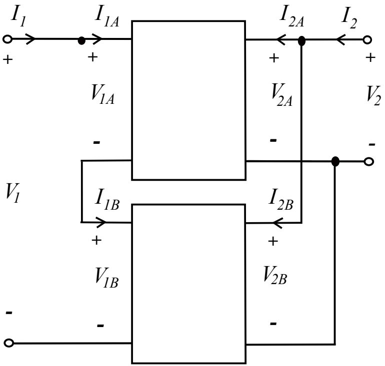
$$Y = Y_A + Y_B$$

Parallel-series connection



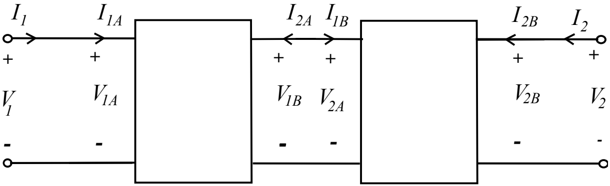
$$g = g_A + g_B$$

Series-parallel connection



$$h = h_A + h_B$$

Cascade connection



$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix}_A + \begin{bmatrix} A & B \\ C & D \end{bmatrix}_B$$

Good to read : `http://en.wikipedia.org/wiki/Two-port_network`

Common emitter

$$v_{be} = f(i_b, v_{ce}) \;\; i_c = g(i_b, v_{ce}$$

small signal analysis:

$$v_{be} = \frac{v_{be}}{i_b} i_b + \frac{v_{be}}{v_{ce}} v_{ce} = h_i i_b + h_r i_b$$

$$i_c = \frac{i_c}{i_b} i_b + \frac{i_c}{v_{ce}} v_{ce} = h_f i_b + h_o v_{ce}$$

- h_i input impedance with $v_{ce} = 0$. This is AC resistance between base and emitter, the reciprocal of the slope of the current-voltage curve of the input characteristics.
- h_r reverse transfer voltage ratio with $i_b = 0$. In general is small and can be ignored.
- h_f forward transfer current ratio or current amplification factor with $v_{ce} = 0$.
- h_o output admittance with $i_b = 0$. It is slope of the current-voltage curve in the output characteristics. In general is small and can be ignored.

Reciprocal Network

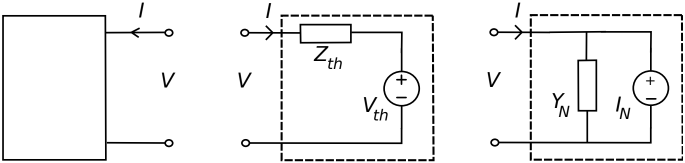
Reciprocity theorem For a reciprocal two-port N, the following relationship holds for each associated two-port representation which exists:

$$z_{12} = z_{21}$$
$$y_{12} = y_{21}$$
$$h_{12} = -h_{21}$$
$$g_{12} = -g_{21}$$

READ : PROOF OF THE RECIPROCITY THEOREM (Chua's book page: 776)

- A gyrator ($i_1 = Gv_2$ and $i_2 = -Gv_1$) is not reciprocal two-port
- An ideal transformer ($v_1 = nv_2$ and $i_2 = ni_1$) is a reciprocal two-port.

Thevenin - Norton Equivalent Circuits



Driving-point characteristic of Thevenin equivalent circuit is defined by

$$V = Z_{\mathrm{th}}(s)I + V_{\mathrm{th}}(s)$$