Circuit and System Analysis
EEF 232E

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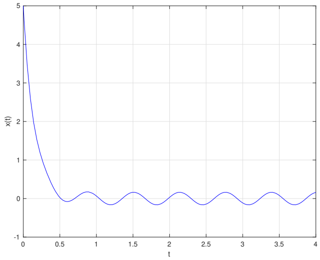
Outline I

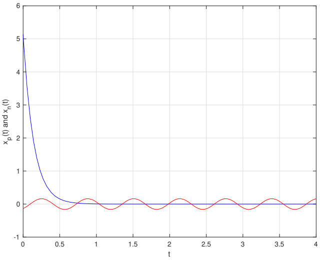
1. Sinusoidal Steady-State Analysis
   - Phasor analysis
   - Properties of phasors
   - Representation of state-space equations
   - Transfer function

Analitic Solution of State-space Equation

$$x(t) = \underbrace{\Phi(t)x(0) - \Phi(t)x_p(0)}_{\text{Natural Response}} + \underbrace{x_p(t)}_{\text{Forced reponse}}$$

Example: $\dot{x} + 7x = 2\sin 10t$ and $x(0) = 5$





$$x(t) = \underbrace{5e^{-7t}}_{\text{zero-input}} + \underbrace{\frac{20}{149}e^{-7t} + \frac{14}{149}\sin(10t) - \frac{20}{149}\sin(10t + \frac{\pi}{2})}_{\text{zero-state}}$$

Sinusoidal Steady-State Analysis

If a system is asymptotically stable

$$\lim_{t \to \infty} \Phi(t) = 0$$

Forced response is the complete response!

$$x(t) = \underbrace{\Phi(t)x_0 - \Phi(t)x_p(t_0)}_{=0} + x_p(t)$$

Sinusoidal steady-state behavior

Sinusoidal steady-state behavior of linear time-invariant circuits when the circuits are driven by one or more sinusoidal sources at some frequency w and when, after all "transients" have died down, all currents and voltages are sinusoidal at frequency w.

Electric Circuits, James W. Nilsson and Susan A. Riedel, Ch.9 and 10

Mathematical Info

$$x(t) = A_m e^{j(wt+\theta)}.$$

where $j = \sqrt{-1}$, A_m ($|X|$) is called the magnitude of $x(t)$ and θ ($\angle$) is called the phase of $x(t)$.
Rectangular representation of complex $x(t)$ is

$$x(t) = A_m \cos(wt + \theta) + jA_m \sin(wt + \theta)$$

Real part

$$A_m \cos(wt + \theta) = \mathfrak{Re}\{x(t)\}$$

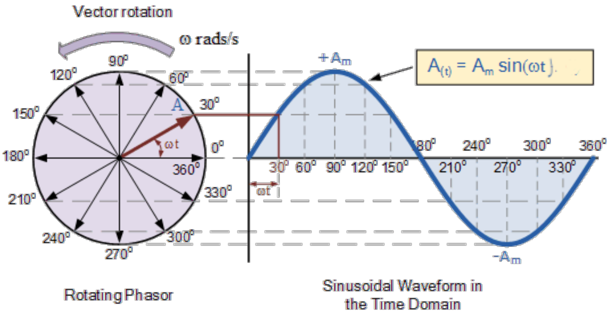
Imaginary part

$$A_m \sin(wt + \theta) = \mathfrak{Im}\{x(t)\}$$

from Euler's identity.
$\cos(wt) = \sin(wt + \frac{\pi}{2})$ and $\sin(wt) = \cos(wt - \frac{\pi}{2})$

Sinusoidal Steady-State Analysis

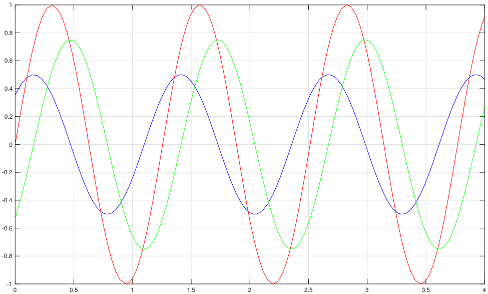
$$x(t) = A_m \sin(wt) = \mathfrak{Im}\{A_m e^{j(wt)}\}$$



$$x(t) = A_m \sin(wt + \theta) = \mathfrak{Im}\{A_m e^{j(wt+\theta)}\}$$

See: Gif image: https://en.wikipedia.org/wiki/File:Unfasor.gif

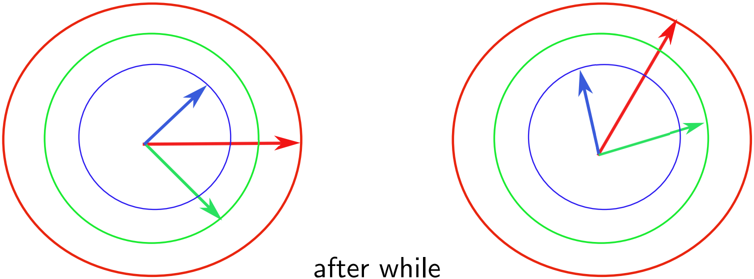
$$\sin(5t), 0.5\sin(5t+\frac{\pi}{4}), 0.75\sin(5t-\frac{\pi}{4})$$



$$\Im m\{1e^{j(5t)}\}, \Im m\{0.5e^{j(5t+\frac{\pi}{4})}\}, \Im m\{0.75e^{j(5t-\frac{\pi}{4})}\}$$

Sinusoidal Steady-State Analysis

$$\mathfrak{Im}\{1e^{j(5t)}\}, \mathfrak{Im}\{0.5e^{j(5t+\frac{\pi}{4})}\}, \mathfrak{Im}\{0.75e^{j(5t-\frac{\pi}{4})}\}$$



after while

We can also use cos function !

$$x(t) = A_m \cos(wt + \theta) = \mathfrak{Re}\{A_m e^{j(wt+\theta)}\}$$

The idea is to associate with each sine wave (of voltage or current) a complex number called the phasor.

$x(t)$ is a complex variable and in polar coordinate

$$x(t) = X_m e^{j(wt+\theta)}.$$

The quantity (**phasor**)

$$X = X_m e^{j\theta}$$

is a complex number that carries the amplitude and phase angle of the given sinusoidal function.This complex number is by definition the **phasor** representation of the given sinusoidal function.

Using phasor representation, a complex variable is given

$$x(t) = X e^{wtj}$$

Examples: Electric Circuits, James W. Nilsson and Susan A. Riedel, pp. 334

Example

Phasor of the sinusoidal function

$$x(t) = 110\sqrt{2}\cos(wt + \frac{\pi}{2})$$

using

$$x(t) = \mathfrak{Re}\{\underbrace{110\sqrt{2}e^{j\frac{\pi}{2}}}_{\text{phasor}} e^{jwt}\}$$

is obtain

$$X = 110\sqrt{2}e^{j\frac{\pi}{2}}$$

$X_m = 100\sqrt{2}$ (or $|X| = 100\sqrt{2}$), $\theta = \frac{\pi}{2}$ (or $\angle\frac{\pi}{2}$). $|X|_{\text{rms}} = 100$
If you like to use $\mathfrak{Im}$:

$$x(t) = 110\sqrt{2}\sin(wt + \pi) = \mathfrak{Im}\{\underbrace{110\sqrt{2}e^{j\pi}}_{\text{phasor}} e^{jwt}\}$$

then $X = 110\sqrt{2}e^{j\pi}$.
Decide one ($\mathfrak{Im}$ or $\mathfrak{Re}$) and obtain phasors, use it for back to t-domain!

Example

Phasor is a complex number therefore remember that:
Lets $Z \in \mathbb{C}$, in rectangular coordinate $Z = \sigma + jw$, in polar coordinate its magnitude $Z_m = \sqrt{\sigma^2 + w^2}$ and its phase $\theta = \mathrm{arctan}(\frac{w}{\sigma})$

$$3 + 4j = 5e^{j0.927} = 5\cos(0.927) + j5\sin(0.927)$$

$$5 = \sqrt{3^2 + 4^2}, \quad \arctan\frac{4}{3} = 0.927\mathrm{randian} = 53.113188^\circ$$

$$1 + j = \sqrt{2}e^{j\frac{\pi}{4}} = \sqrt{2}\cos(\frac{\pi}{4}) + j\sqrt{2}\sin(\frac{\pi}{4}))$$

$$\sqrt{2} = \sqrt{1^2 + 1^2} \quad \arctan\frac{1}{1} = \frac{\pi}{4}\text{randian} = 45^\circ$$

$$5e^{j0.927} + \sqrt{2}e^{j0.785} =?$$

$$3 + 4j + 1 + j = 4 + 5j = 6.403e^{j0.896}$$

$$(3 + 4j) \times (1 + j) =?$$

$$5e^{j0.927} \times \sqrt{2}e^{j0.785} = 5\sqrt{2}e^{j1.71}$$

Properties of phasors

- $$y(t) = \alpha_1 x_1(t) + \alpha_2 x_2(t)$$

  $$Y = \alpha_1 X_1 + \alpha_2 X_2$$

- X is phasor of $x(t)$

  $$y(t) = \frac{d^n x(t)}{dt^n}$$

  then phasor of $y(t)$

  $$Y = (jw)^n X.$$

- $$\frac{d}{dt}\{X_m e^{\theta j} e^{jwt}\} = \{\frac{d}{dt} X_m e^{\theta j} e^{jwt}\} = \{\underbrace{jwAX_m e^{\theta j}}_{\text{phasor}} e^{jwt}\}$$

- A are B phasor
  - If $\mathfrak{Re}\{Ae^{jwt}\} = \mathfrak{Re}\{Be^{jwt}\}$ then $A = B$.
  - If $A = B$ then $\mathfrak{Re}\{Ae^{jwt}\} = \mathfrak{Re}\{Be^{jwt}\}$.

Phasor & State-space equation

Lets find the sinusoidal particular solution (Xe^{jwt}) of linear time invariant state equation

$$\dot{x} = Ax + Bu$$

for a sinusoidal input Ue^{jwt}. Substituting the solution and input

$$j(w)X = AX + BU$$

The sinusoidal solution is then

$$X = (jwI - A)^{-1}BU$$

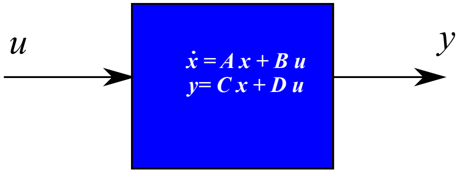
The solution is defined for $\det(jwI - A) \neq 0$ which means $jw \neq \lambda$.Input frequency is equal the natural frequency.

Transfer function

A linea time invariant single input single output system $(u, y \in R)$ is defined by

$$\dot{x} = Ax + Bu$$
$$y = Cx + Du$$

what if $u \in R^m$, $y \in R^l$ ...



Using phasors, the output of the system

$$Y = \underbrace{\left(C(jwI - A)^{-1}B + D\right)}_{\text{Transfer function}} U$$

Transfer function

$$H(jw) = \left(C(jwI - A)^{-1}B + D\right)$$

from input U to output Y.

Example

$$\frac{dx}{dt} = \begin{bmatrix} -\sqrt{2} & -1 \\ 1 & 0 \end{bmatrix} x + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u(t); \quad y = [0\ 1]x$$

Transfer function (which is a function of w !)

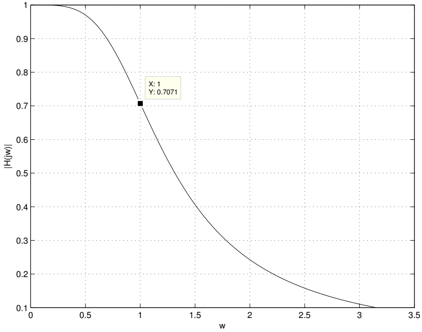
$$H(jw) = \frac{1}{1 - w^2 + j\sqrt{2}w}$$

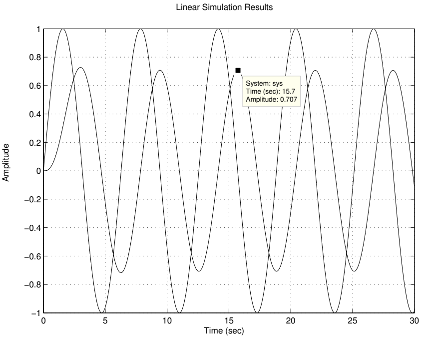
For the input $u(t) = \sin(wt)$(phasor of input signal $E = 1$) phasor of the output is

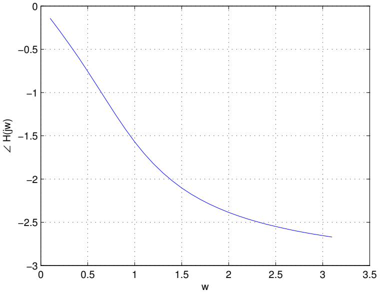
$$Y = H(j1)E = \frac{-j}{\sqrt{2}} = \frac{1}{\sqrt{2}} e^{-j\pi/2}$$

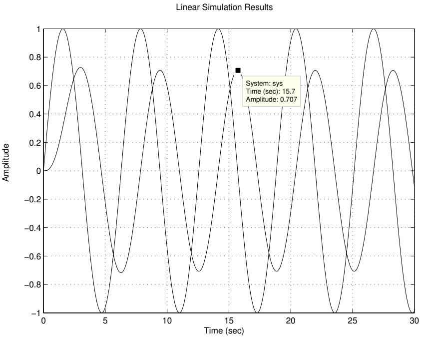
Hence output signal is

$$y(t) = \frac{1}{\sqrt{2}} \sin(t - \pi/2)$$


X: 1
Y: 0.7071
|H(jw)|
w
0
0.5
1
1.5
2
2.5
3
3.5
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1


Linear Simulation Results
System: sys
Time (sec): 15.7
Amplitude: 0.707
Amplitude
Time (sec)
0
5
10
15
20
25
30
−1
−0.8
−0.6
−0.4
−0.2
0
0.2
0.4
0.6
0.8
1


∠ H(jw)
w
0
−0.5
−1
−1.5
−2
−2.5
−3
0
0.5
1
1.5
2
2.5
3
3.5


Linear Simulation Results
Amplitude
Time (sec)
System: sys
Time (sec): 15.7
Amplitude: 0.707
1
0.8
0.6
0.4
0.2
0
−0.2
−0.4
−0.6
−0.8
−1
0
5
10
15
20
25
30

Analitic Solution of State-space Equation

Example: $\dot{x} + 7x = 2\sin 10t$ and $x(0) = 5$

$$\dot{x} + 7x = 2\sin 10t$$

, Remember $w = 10$

$$jwX + 7X = 2$$
$$j10X + 7X = 2$$
$$(7 + j10)X = 2$$

$$X = \frac{2}{7+10j} = \frac{2}{149}(7-10j) = \frac{14}{149} - j\frac{20}{149}$$

$$x(t) = \frac{14}{149}\sin(10t) - \frac{20}{149}\sin(10t + \frac{\pi}{2})$$