Circuit and System Analysis
EEF 232E

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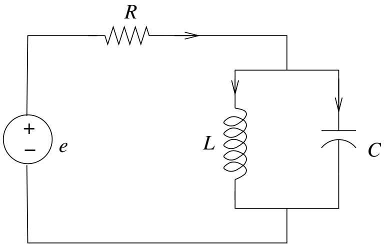
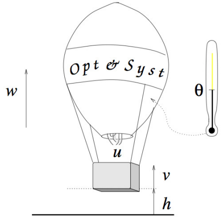
Outline I

1. Mathematical Systems Theory
   - Modelling an Engineering System
   - Linear and Time-invariant System
   - State-space Equation
   - Analitic Solution of State-space Equation
   - Stability



State Equation of the circuit;

$$\frac{d}{dt}\begin{bmatrix} V_C \\ i_L \end{bmatrix} = \begin{bmatrix} -\frac{1}{RC} & -\frac{1}{C} \\ \frac{1}{L} & 0 \end{bmatrix}\begin{bmatrix} V_C \\ i_L \end{bmatrix} + \begin{bmatrix} \frac{1}{RC} \\ 0 \end{bmatrix} e$$

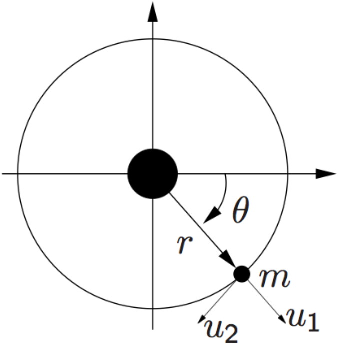


State Equation of The Hot Air Balloon;

$$\frac{d}{dt}\begin{bmatrix}\theta \\ v \\ h\end{bmatrix} = \begin{bmatrix} -\frac{1}{\alpha} & 0 & 0 \\ \sigma & -\frac{1}{\beta} & \frac{1}{\beta} \\ 0 & 1 & 0 \end{bmatrix}\begin{bmatrix}\theta \\ v \\ h\end{bmatrix} + \begin{bmatrix}1 \\ 0 \\ 0\end{bmatrix} u$$

$\theta =$ temperature, $u =$ heating, $v =$ vertical velocity, $h =$ height, $w =$ vertical wind velocity

State Equations



Satellite Control;

$$\ddot{r}(t) = r(t)\dot{\theta}^2(t) - \frac{k}{r^2(t)} + u_1(t)$$
$$\ddot{\theta}(t) = \frac{\dot{\theta}(t)\dot{r}(t)}{r(t)} + \frac{1}{r(t)}u_2(t)$$

State Equations

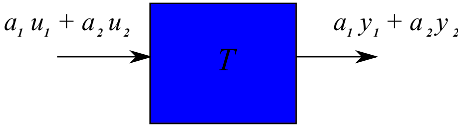
A DC motor consists of an electromagnet made by winding wires around a core placed in a magnetic field made with permanent magnets or electromagnets. When current flows through the wires, the core spins.

$$
\begin{aligned}
L\dot{i}(t) &= v(t) - Ri(t) - k_b w(t) \\
I\dot{w}(t) &= k_T i(t) - \kappa w(t) - \tau(t)
\end{aligned}
$$

where w Angular Velocity, k_b back electromagnetic force constant, k_T motor torque constant, μ is the kinetic friction of the motor, and τ is the torque applied by the load.

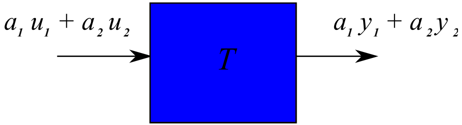
See : http://leeseshia.org/download.html , page 203 (motor-RL-model.m)

Linear System



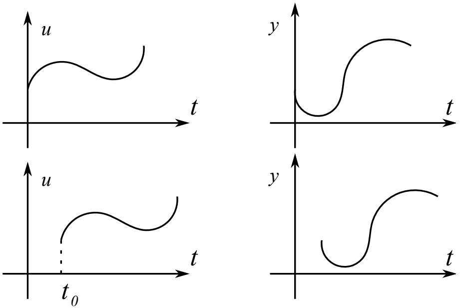
- T is the system
- u is the input.
- y is the output

Linear System



Here $a_1, a_2 \in R$ and Input u_1 gives response y_1, Input u_2 gives response y_2, Input $a_1 u_1 + a_2 u_2$ gives response $a_1 y_1 + a_2 y_2$.

Time-invariance



This implies that the dynamics do not change over time.

$$\dot{x} = Ax + Bu, \quad x(t_0) = x_0$$

where the state variable $x \in R^n$, input $u \in R^m$ and $A \in R^{n \times n}$, $B \in R^{n \times m}$

Output $y \in R^l$

$$y = Cx + Du$$

where $C \in R^{l \times n}$ and $D \in R^{l \times m}$.

Analitic Solution of State-space Equation

$$x(t) = \underbrace{x_{zi}(t)}_{\text{zero-input response}} + \underbrace{x_{zs}(t)}_{\text{zero-state response}}$$

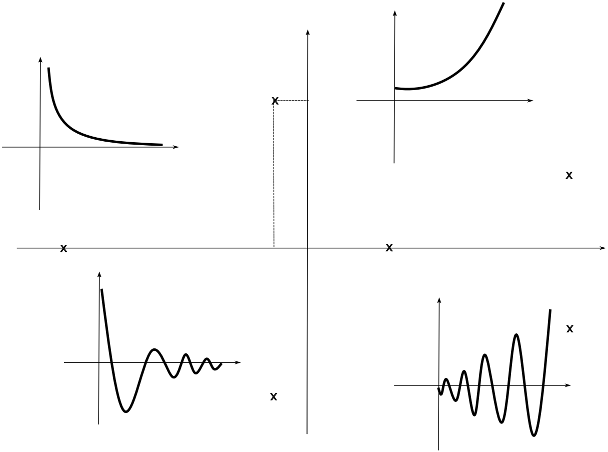
$$x(t) = \underbrace{\Phi(t)x(0)}_{\text{zero-input response}} + \underbrace{\int_0^t \Phi(t-\tau)Bu(\tau)d\tau}_{\text{zero-state response}}$$

$$\Phi(t) = \begin{bmatrix} v_1 e^{\lambda_1 t} & v_2 e^{\lambda_2 t} & \dots & v_n e^{\lambda_n t} \end{bmatrix} \begin{bmatrix} v_1 e^{\lambda_1 t_0} & v_2 e^{\lambda_2 t_0} & \dots & v_n e^{\lambda_n t_0} \end{bmatrix}^{-1}$$

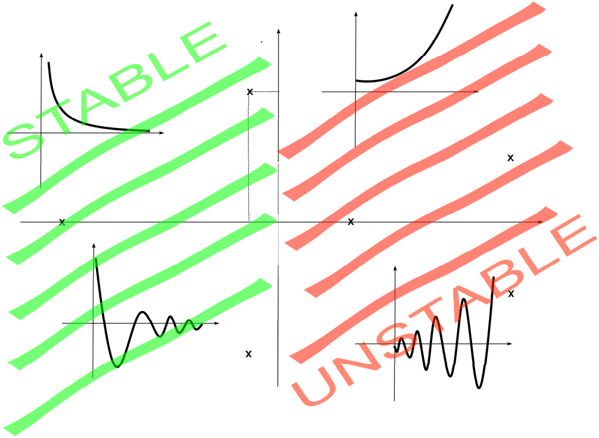
Question : What happen if $\lim_{t\to\infty} \Phi(t)$

Hint : λ which is eigenvalue !

Analitic Solution of State-space Equation



Remember form Lineer Algebra $\lambda = \sigma + jw \in \mathbb{C}$ and complex numbers can be visualized geometrically as points in the complex plane.
Remember : Euler's formula $e^{jx} = \cos x + j \sin x$


STABLE
UNSTABLE
x
x
x
x
x
x
x

$$x(t) = \underbrace{\Phi(t)x(0)}_{\text{zero-input response}} + \underbrace{\int_0^t \Phi(t-\tau)Bu(\tau)d\tau}_{\text{zero-state response}}$$

$$= \underbrace{\Phi(t)x(0) - \Phi(t)x_p(0)}_{\text{Natural Response}} + \underbrace{x_p(t)}_{\text{Forced response}}$$

If $\lambda < 0$ system is called "asymptotically stable" which means $(e^{\lambda t})$

Then :

$$x(t) = \underbrace{x_p(t)}_{\text{Forced response}}$$

What if $\lambda = jw$

$$\frac{d}{dt}\begin{bmatrix} i_L \\ V_C \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} i_L \\ V_C \end{bmatrix}$$

Solve

$$\det\left\{\lambda I - \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\right\} = \lambda^2 + 1$$

thus eigenvalues are $\lambda_1 = j$, $\lambda_2 = -j$ and corresponding eigenvectors are $[j \;\; -1]^T$, $[-j \;\; -1]^T$.

$$\phi(t) = \begin{bmatrix} je^{jt} & -je^{-jt} \\ -e^{jt} & -e^{-jt} \end{bmatrix}\begin{bmatrix} j & -j \\ -1 & -1 \end{bmatrix}^{-1} = \begin{bmatrix} \cos(t) & \sin(t) \\ -\sin(t) & \cos(t) \end{bmatrix}$$

$$x_{zi}(t) = \begin{bmatrix} \cos(t) & \sin(t) \\ -\sin(t) & \cos(t) \end{bmatrix} \underbrace{\begin{bmatrix} 0 \\ 1 \end{bmatrix}}_{\text{Initial Cond.}} = \begin{bmatrix} \sin(t) \\ \cos(t) \end{bmatrix}$$

Only stable (not asymptotically stable)