MATHEMATICS 101E FINAL EXAM 2 January 2014

QUESTION 1 The blanks below will be filled by students. (Except the score)

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For the solution of this question please use only the front face and if necessary the back face of this page.

[12pts] **a)** $\int \sinh(\sqrt[3]{x})\,dx = ?$

[13pts] **b)** $\int \frac{x^4+4x+2}{x^3+2x}\,dx = ?$

Solution :

a) $\int \sinh(\sqrt[3]{x})\,dx = \int 3t^2 \sinh t\,dt$ $\left[\begin{array}{ll} u=3t^2, & dv=\sinh t\,dt \\ du=6t\,dt & v=\cosh t \end{array}\right]$

$\left(\begin{array}{l} x=t^3 \\ dx=3t^2dt \end{array}\right)$ $= 3t^2\cosh t - 6\int t\cosh t\,dt$ $\left[\begin{array}{ll} u=t, & dv=\cosh t\,dt \\ du=dt & v=\sinh t \end{array}\right]$

$= 3t^2\cosh t - 6\left[t\sinh t - \int \sinh t\,dt\right]$

$= 3t^2\cosh t - 6t\sinh t + 6\cosh t + C$ $(t=x^{1/3})$

$= 3x^{2/3}\cosh x^{1/3} - 6x^{1/3}\sinh x^{1/3} + 6\cosh x^{1/3} + C.$

b) $I = \int \frac{x^4+4x+2}{x^3+2x}\,dx = \int\left(x + \frac{-2x^2+4x+2}{x^3+2x}\right)dx$

$$\begin{array}{r|l} x^4+4x+2 & x^3+2x \\ \underline{-x^4 \mp 2x^2} & x \\ -2x^2+4x+2 & \end{array}$$

$= \int x\,dx + \int \frac{-2x^2+4x+2}{x(x^2+2)}\,dx$

$\cdot\ \frac{-2x^2+4x+2}{x(x^2+2)} = \frac{A}{x} + \frac{Bx+C}{x^2+2} = \frac{1}{x} + \frac{-3x+4}{x^2+2}.$

$(A=1,\ B=-3,\ C=4)$

$\Rightarrow I = \int x\,dx + \int \frac{1}{x}\,dx + \int \frac{-3x}{x^2+2}\,dx + 4\int \frac{dx}{x^2+2}$

$= \frac{x^2}{2} + \ln|x| - \frac{3}{2}\ln(x^2+2) + \frac{4}{\sqrt{2}}\tan^{-1}\left(\frac{x}{\sqrt{2}}\right) + C.$

QUESTION 2 The blanks below will be filled by students. (Except the score)

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[12pts] **a)** By using the definition of the definite integral, find the following limit

$$\lim_{n\to\infty}\frac{1}{n}\left(e^{\frac{1}{2n}}+e^{\frac{2}{2n}}+\cdots+e^{\frac{n-1}{2n}}\right)$$

[13pts] **b)** Find the area of the region outside the curve $\mathbf{r}=\mathbf{1}-\cos\theta$ and inside the curve $\mathbf{r}=\sqrt{\mathbf{3}}\,\sin\theta$.

Solution :

a) $\lim\limits_{n\to\infty}\frac{1}{n}\left(e^{\frac{1}{2n}}+e^{\frac{2}{2n}}+\cdots+e^{\frac{n-1}{2n}}\right)=\lim\limits_{n\to\infty}\frac{1}{n}\left\{-1+1+e^{\frac{1}{2n}}+e^{\frac{2}{2n}}+\cdots+e^{\frac{n-1}{2n}}\right\}$

$=\lim\limits_{n\to\infty}\left(\frac{-1}{n}\right)+\lim\limits_{n\to\infty}\frac{1}{n}\left\{1+e^{\frac{1}{2n}}+e^{\frac{2}{2n}}+\cdots+e^{\frac{n-1}{2n}}\right\}$

$=\lim\limits_{n\to\infty}\sum\limits_{k=1}^{n}\frac{1}{n}e^{\frac{k-1}{2n}}$

$c_k=a+(k-1)\Delta x_k=\frac{k-1}{n}$

$\Delta x_k=\frac{b-a}{n}=\frac{1}{n}\Rightarrow b-a=1$

$\Rightarrow a=0\ ,\ b=1$

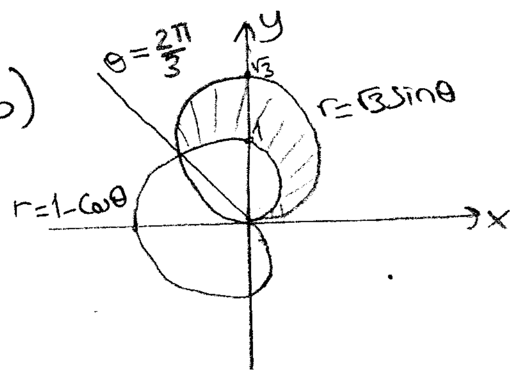
$$\boxed{\lim_{n\to\infty}\sum_{k=1}^{n}f(c_k)\Delta x_k=\int_a^b f(x)\,dx}$$

$f(c_k)=e^{\frac{c_k}{2}}\Rightarrow\boxed{f(x)=e^{\frac{x}{2}}}$

$\Rightarrow\lim\limits_{n\to\infty}\frac{1}{n}\left(e^{\frac{1}{2n}}+e^{\frac{2}{2n}}+\cdots+e^{\frac{n-1}{2n}}\right)=\int_0^1 e^{\frac{x}{2}}\,dx$

$=2e^{\frac{x}{2}}\Big|_0^1=2(\sqrt{e}-1).$

b)



$\sqrt{3}\sin\theta=1-\cos\theta$

$3\underbrace{\sin^2\theta}_{(1-\cos^2\theta)}=1-2\cos\theta+\cos^2\theta$

$4\cos^2\theta-2\cos\theta-2=0$

$2\cos^2\theta-\cos\theta-1=(2\cos\theta+1)(\cos\theta-1)=0$

$\Rightarrow\theta=0\ ,\ \theta=\frac{2\pi}{3}$

$\Rightarrow A=\frac{1}{2}\int_{\theta=0}^{\frac{2\pi}{3}}\left[(\sqrt{3}\sin\theta)^2-(1-\cos\theta)^2\right]d\theta=\frac{1}{2}\int_0^{2\pi/3}\Big(2\cos\theta-4\underbrace{\cos^2\theta}_{\frac{1+\cos2\theta}{2}}+2\Big)\,d\theta$

$=\frac{1}{2}\left[2\sin\theta-2\left(\theta+\frac{\sin2\theta}{2}\right)+2\theta\right]\Big|_0^{\frac{2\pi}{3}}=\left(\sin\theta-\frac{1}{2}\sin2\theta\right)\Big|_0^{\frac{2\pi}{3}}=\underline{3\sqrt{3}}.$

QUESTION 3 The blanks below will be filled by students. (Except the score)

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[12pts] **a**) Evaluate the length of the curve $\mathbf{x} = \cos \mathbf{2t}$, $\mathbf{y} = \mathbf{2t} + \sin \mathbf{2t}$ for $\mathbf{0} \leq \mathbf{t} \leq \frac{\pi}{\mathbf{2}}$.

[13pts] **b**) Write the integral that calculates the volume of the solid generated by revolving the region enclosed by the curves $\mathbf{y} = \mathbf{x}^2 + \mathbf{1}$, $\mathbf{x} = \mathbf{2}\sqrt{\mathbf{y}}$ and the lines $\mathbf{x} = \mathbf{3} - \mathbf{y}$, $\mathbf{x} = \mathbf{0}$ in the first quadrant about the line $\mathbf{y} = -\mathbf{1}$. (Do not evaluate the integral.)

Solution :

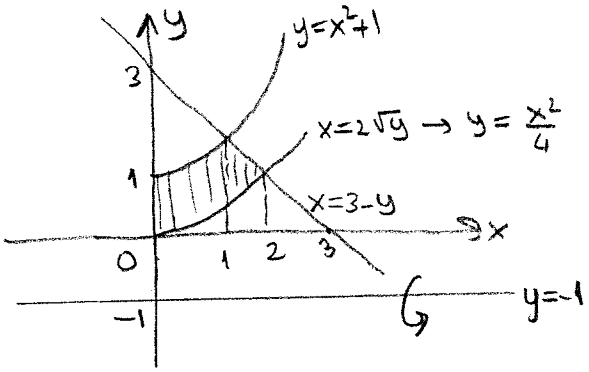
a) $\left.\begin{aligned} x &= \cos 2t \\ y &= 2t + \sin 2t \end{aligned}\right\} \Rightarrow \left.\begin{aligned} \frac{dx}{dt} &= -2\sin 2t \\ \frac{dy}{dt} &= 2 + 2\cos 2t \end{aligned}\right\} \Rightarrow \left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 = 8(1+\cos 2t)$

$$L = \int_0^{\frac{\pi}{2}} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt = \int_0^{\frac{\pi}{2}} \sqrt{8(1+\cos 2t)}\, dt = \int_0^{\frac{\pi}{2}} \sqrt{8(\cancel{1} + 2\cos^2 t - \cancel{1})}\, dt$$

$$= \int_0^{\frac{\pi}{2}} 4|\cos t|\, dt = 4\int_0^{\frac{\pi}{2}} \cos t\, dt \qquad \left(0 \leq t \leq \frac{\pi}{2} \to \cos t \geq 0\right)$$

$$= 4 \sin t \Big|_0^{\frac{\pi}{2}} = 4.$$

b)



$x = 2\sqrt{y}$, $x = 3-y$

$2\sqrt{y} = 3-y$

$4y = 9 - 6y + y^2$

$y^2 - 10y + 9 = 0$

$y = 1$, $y = 9$

$\boxed{x = 2}$, $x = 6$

$y = x^2+1$, $y = 3-x$

$x^2 + 1 = 3 - x$

$x^2 + x - 2 = 0$

$\boxed{x = 1}$, $x = -2$

$$V_{y=-1} = \pi \int_a^b (y+1)^2\, dx$$

$$V_{y=-1} = \pi \int_0^1 \left\{ \left[(x^2+1)+1\right]^2 - \left(\frac{x^2}{4}+1\right)^2 \right\} dx + \pi \int_1^2 \left\{ \left[(3-x)+1\right]^2 - \left(\frac{x^2}{4}+1\right)^2 \right\} d$$

$$= \pi \int_0^1 \left(\frac{15}{16}x^4 + \frac{7}{2}x^2 + 3\right) dx + \pi \int_1^2 \left(\frac{x^2}{2} - \frac{x^4}{16} - 8x + 15\right) dx .$$

QUESTION 4 The blanks below will be filled by students. (Except the score)

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[12pts] **a)** $\displaystyle\lim_{x\to 0}\frac{2e^x-2+\int_{2x}^{0}e^{\sin^{-1}t}\,dt}{1-\cos x}=?$

[7pts] **b)** Investigate the convergence or divergence of the improper integral $\displaystyle\int_1^\infty\frac{e^{-x}}{\sqrt{1+x^2}}\,dx$.

[6pts] **c)** Evaluate the improper integral $\displaystyle\int_0^1\frac{dx}{\sqrt{4x-x^2}}$.

Solution :

a) $$\lim_{x\to 0}\frac{2e^x-2+\int_{2x}^{0}e^{\sin^{-1}t}\,dt}{1-\cos x}\overset{\frac{0}{0}}{=}\lim_{x\to 0}\frac{2e^x-2e^{\sin^{-1}(2x)}}{\sin x}\ \left(\frac{0}{0}\right)$$

$$=\lim_{x\to 0}\frac{2e^x-2\frac{2}{\sqrt{1-4x^2}}e^{\sin^{-1}(2x)}}{\cos x}=-2.$$

b) $$\int_1^\infty\frac{e^{-x}}{\sqrt{1+x^2}}\,dx\ ,\quad f(x)=\frac{e^{-x}}{\sqrt{1+x^2}}=\frac{1}{e^x\sqrt{1+x^2}}\ ,\quad g(x)=\frac{1}{x^2}$$

$$\Rightarrow\lim_{x\to\infty}\frac{f(x)}{g(x)}=\lim_{x\to\infty}\frac{x^2}{e^x\sqrt{1+x^2}}=\left(\lim_{x\to\infty}\frac{x}{e^x}\right)\underbrace{\left(\lim_{x\to\infty}\frac{x}{x\sqrt{\frac{1}{x^2}+1}}\right)}_{=1}\ \left(\frac{\infty}{\infty}\right)$$

$$=\lim_{x\to\infty}\frac{1}{e^x}=0=L$$

$L=0$ and $\int_1^\infty g(x)\,dx=\int_1^\infty\frac{1}{x^2}\,dx$ $(p=2>1)$ converges $\Rightarrow\int_1^\infty f(x)\,dx$ converges

c) $$\int_0^1\frac{dx}{\sqrt{4x-x^2}}=\lim_{a\to 0^+}\int_a^1\frac{dx}{\sqrt{4-(x-2)^2}}=\lim_{a\to 0^+}\left(\sin^{-1}\frac{x-2}{2}\right)\Big|_a^1$$

$$\left(\begin{matrix}x-2=\sin t\\ dx=\cos t\,dt\end{matrix}\right)\qquad=\lim_{a\to 0^+}\left[\sin^{-1}\left(-\tfrac{1}{2}\right)-\sin^{-1}\left(\tfrac{a-2}{2}\right)\right]$$

$$=-\frac{\pi}{6}-\left(-\frac{\pi}{2}\right)=\frac{\pi}{3}.$$