Quantitative Prediction of the Effects of Mistuning Arrangement on Resonant Response of a Practical Turbine Bladed Disc

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Abstract

The effects of blade mistuning have been studied extensively in the past although most of the research efforts have been directed towards understanding its **qualitative** aspects. Addressing the **quantitative** aspects of the mistuning problem, especially for industrial designs, has been somewhat limited in the past, mainly due to the need for using reduced-order and thus simplified models in such analyses. In other words, it has been difficult to translate previous mistuning results directly for high-cycle fatigue predictions for a given bladed-disc.

A new method for the dynamic analysis of mistuned bladed discs is presented. The method is based on exact calculation of the response of the mistuned system using response levels of the tuned assembly and a modification matrix constructed from the Frequency Response Function (FRF) matrix of the tuned system and a matrix describing the mistuning. The main advantages of the method are its efficiency and accuracy, which allows the use of large finite element models of practical bladed disc assemblies in parametric studies of mistuning effects on vibration amplitudes. A new method of calculating the FRF matrix of the tuned system using a sector model is also developed so as to improve the efficiency of the method further, making the proposed method a very attractive tool for mistuning studies of practical design.

1. Introduction

Small variations of individual blade characteristics in a bladed assembly inevitably arise during blade manufacturing and assembling processes. It is well known that the dynamic response of a mistuned bladed disc can be altered significantly from that of its tuned counterpart so that the response of amplitudes of individual blades may vary widely within the same assembly (Dye and Henry, 1969), (Ewins, 1969) and (Whitehead, 1966). This situation has posed a very important practical problem for a long time and efforts of many investigators have been, and continue to be, devoted to predicting and controlling these mistuning effects. Surveys of recent studies of mistuning are given in (Ewins, 1991) and (Slater, 1999).

The current trend in the turbomachinery industry is to avoid the various simplifying assumptions during the modelling process and to seek quantitative answers to questions related to mistuning. Using realistic finite element models for the complete bladed disc, including mistuning effects is, however, still too expensive in spite of the advances in finite element modelling and computer hardware during recent years. The common practice in industry is to make use of the cyclic symmetry properties of the tuned system to obtain the natural frequencies and mode shapes of the whole tuned bladed disc assemblies. Although very desirable, there is no method readily available to use the models based on cyclic symmetry for mistuning studies. Attempts at using detailed finite element models for mistuned bladed discs have been made recently. An original approach has been developed in (Bladh et al., 1998) and (Castanier et al. 1997) to reduce the size of the large-scale finite element models of mistuned bladed discs. The developed technique is similar to the Ritz procedure and is based on expansion of the mistuned bladed disc amplitudes into a series of mode shapes calculated for two specially chosen subsystems of the tuned bladed disc. Some investigation of the validity of this technique can be found in reference (Frey, 1998). A different method is proposed in (Yang and Griffin, 1998), where model reduction is carried out by representation of mode shapes as a limited sum of the system modes that are called “nominal” modes by the authors. These can be the modes of the tuned bladed disc or of the bladed disc with a specified mistuning pattern. Both of the above mentioned approaches are dependent on the nominal mode shapes chosen for the expansions of the mistuned system response as well as on the number of these mode shapes.

The present paper introduces a new method for the analysis of forced vibration of mistuned bladed discs. The method is based on an exact relationship between the response levels of the tuned and the mistuned bladed discs, the tuned system being described by a sector model using the cyclic symmetry properties of the assembly. An exact relationship between the tuned and the mistuned system is derived using Woodbury-Sherman-Morrison formula for the inverse of a perturbed matrix (Sherman and Morrison, 1949) and (Woodbury, 1950). This formula was applied in reference (Level et al., 1996) for response reanalysis of a simple linear system and was proposed and used successfully for calculations of modifications in the non-linear analysis of systems with friction dampers in references (Sanliturk et al., 1999). An important feature of the method proposed here is the reduction of the system model to a manageable size without introducing any loss of accuracy during the reduction process, such as is usually incurred when including only a subset of the model degrees of freedom reflecting the interest of the eventual application. Any subset of nodes of interest can be chosen from the whole set of assembly nodes for the mistuning analysis without any adverse effect. Accordingly, the method allows mistuning analysis of bladed discs using large finite element models that are used at present only in the analysis of tuned bladed discs based on cyclic symmetry approach.

The second section of the paper describes an exact relationship between the response amplitudes of the mistuned and the tuned bladed discs as well as how to perform the response calculations for the mistuned system at those co-ordinates where the system is mistuned or amplitudes are controlled. The section explores also how the tuned system can be used during the forced response analysis of a mistuned assembly. An original technique is presented for the calculation of both forced response levels and the Frequency Response Function (FRF) matrix of tuned bladed discs using a sector model. This method allows one to obtain a tuned assembly's response properties under arbitrarily distributed harmonic loads over all bladed disc sectors, a feature essential for the analysis of mistuned bladed discs. Mistuning devices or 'elements' are defined in the third section to facilitate simulation of the different mistuning conditions encountered in practice. An approach to establish a relationship between mistuning elements and blade-alone frequency mistuning is also described. The last section of the paper presents a case study of a mistuned turbine bladed disc with mistuned blades and with damaged blades which have very large deviations in blade-alone natural frequencies.

2. A new method for forced response analysis of mistuned bladed discs

2.1 An exact relationship between the response of tuned and mistuned bladed discs

The equation of motion for forced vibration of a bladed disc such as that shown in Fig.1 can be written in a customary form in the frequency domain as:

$$(\boldsymbol{K}-\omega^2\boldsymbol{M}+i\boldsymbol{D})\boldsymbol{q}=\boldsymbol{Z}(\omega)\boldsymbol{q}=\boldsymbol{f} \qquad (1)$$

where $\boldsymbol{q}$ is a vector of complex response amplitudes for nodal displacements; $\boldsymbol{f}$ is a vector of complex amplitudes of harmonic nodal loads; $\boldsymbol{K}$, $\boldsymbol{M}$ and $\boldsymbol{D}$ are stiffness, mass and structural damping matrices of the system, respectively; $\boldsymbol{Z}(\omega)$ is the dynamic stiffness matrix; ω is excitation frequency; and $i=\sqrt{-1}$. One can also include other terms representing gyroscopic and stiffening effects due to rotation in the dynamic stiffness matrix, if required.

It is proposed here to represent the dynamic stiffness matrix of a mistuned bladed disc as a sum of two matrices; a matrix corresponding to the tuned bladed disc, $\boldsymbol{Z}_0$, and a mistuning matrix, $\Delta\boldsymbol{Z}$, which reflects the deviation from the tuned system. As a result, the matrix equation for the forced response of a mistuned bladed disc (1) can be written as:

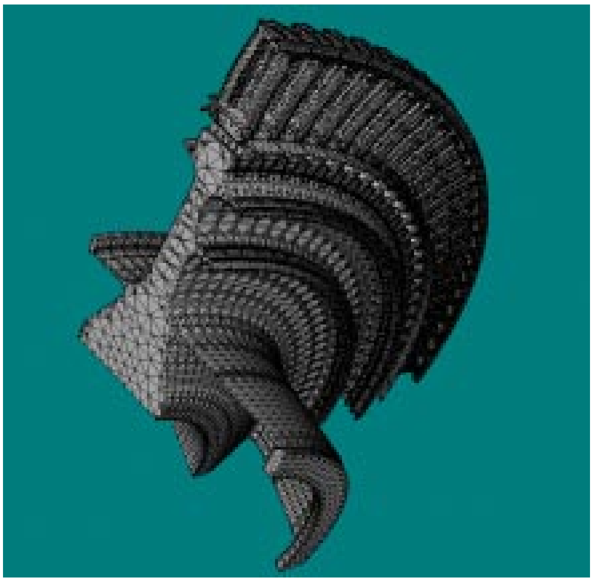


Fig. 1 Finite element model of a bladed disc

$$[\boldsymbol{Z}_0(\omega)+\Delta\boldsymbol{Z}(\omega)]\boldsymbol{q}=\boldsymbol{f} \quad (2)$$

and so the response of the mistuned bladed disc is expressed as:

$$\boldsymbol{q}=[\boldsymbol{Z}_0(\omega)+\Delta\boldsymbol{Z}(\omega)]^{-1}\boldsymbol{f} \quad (3)$$

However, calculation of the response levels by direct solution of Eq.(2), or using the inverse of a matrix as implied by the above equation, is extremely costly, especially for the analysis of a complete mistuned bladed disc assembly when a realistic finite element model is to be used. The major idea in this paper is to obtain the forced response levels for a mistuned bladed disc with high accuracy without the need for:

(i) a matrix inversion;
(ii) a complete finite element model for the whole assembly,
(iii) inclusion of all the degrees of freedom in the response analysis.

It is shown in references (Sherman and Morrison, 1949) and (Woodbury, 1950) that the inverse of the sum of two matrices may be obtained using the so-called Sherman-Morrison-Woodbury identity which relates the inverse of the sum of two matrices to the inverse of one of the summands, transformed with the use of the second summand:

$$(\boldsymbol{Z}_0+\boldsymbol{U}\boldsymbol{V}^T)^{-1}=\boldsymbol{Z}_0^{-1}-\boldsymbol{Z}_0^{-1}\boldsymbol{U}(\boldsymbol{I}+\boldsymbol{V}^T\boldsymbol{Z}_0^{-1}\boldsymbol{U})^{-1}\boldsymbol{V}^T\boldsymbol{Z}_0^{-1} \quad (4)$$

This identity is valid for any matrix $\boldsymbol{Z}_0$ ($N\times N$) and any matrix $\Delta\boldsymbol{Z}=\boldsymbol{U}\boldsymbol{V}^T$, expressed by multiplication of two rectangular matrices $\boldsymbol{U}$ ($N\times n$) and $\boldsymbol{V}$ ($N\times n$) if $\boldsymbol{Z}_0$ and $\boldsymbol{I}+\boldsymbol{V}^T\boldsymbol{Z}_0^{-1}\boldsymbol{U}$ are invertible. Here $\boldsymbol{I}$ is an identity matrix, N is the total number of degrees of freedom for the system considered, and n is number of degrees of freedom where modifications are made and forced response analysed.

For the mistuning problem addressed in this paper, the matrix $\boldsymbol{Z}_0$ is the dynamic stiffness matrix of the tuned bladed disc and the matrix $\Delta\boldsymbol{Z}$ represents the “mistuning matrix” which can itself be represented as a multiplication of two matrices. This representation can be done in various ways, one of which is to collect all nonzero rows into a matrix $\boldsymbol{V}^{\mathrm{T}}$ ($n\times N$) and to construct the matrix $\boldsymbol{U}$ ($N\times n$) using unit vectors corresponding to nonzero rows of the matrix $\boldsymbol{V}$. Inserting Eq.(4) into Eq.(3), one can obtain an expression for the response of the mistuned bladed disc in terms of that of the tuned system and the modification matrices $\boldsymbol{U}$ and $\boldsymbol{V}$ as

$$\begin{aligned}\boldsymbol{q}&=[\boldsymbol{A}_0-\boldsymbol{A}_0\boldsymbol{U}(\boldsymbol{I}+\boldsymbol{V}^T\boldsymbol{A}_0\boldsymbol{U})^{-1}\boldsymbol{V}^T\boldsymbol{A}_0]\boldsymbol{f}=\\&=\boldsymbol{q}_0-[\boldsymbol{A}_0\boldsymbol{U}(\boldsymbol{I}+\boldsymbol{V}^T\boldsymbol{A}_0\boldsymbol{U})^{-1}\boldsymbol{V}^T]\boldsymbol{q}_0\end{aligned} \quad (5)$$

where $\boldsymbol{A}_0(\omega)=[\boldsymbol{Z}_0(\omega)]^{-1}$ is the receptance (FRF) matrix for the tuned bladed disc, and $\boldsymbol{q}_0$ is a vector representing the amplitudes of the tuned assembly due to the external force vector, $\boldsymbol{f}$.

An effective method for calculation the FRF matrix of the tuned bladed disc is presented in reference by Petrov et al. [8, 9], where natural frequencies and mode shapes obtained from a sector model (Fig.2a) of the tuned assembly are used for generation of the FRF matrix for the whole assembly.

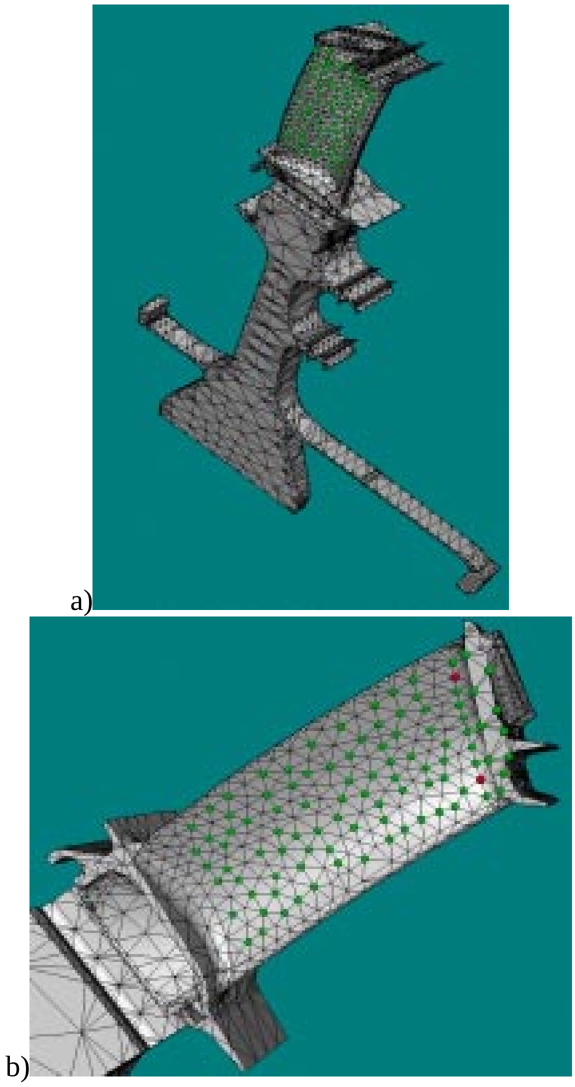


Fig. 2 Finite element sector model (a) and active nodes (b)

The expression for the FRF matrix of the whole bladed-disc assembly derived in reference [10] has the following form:

$$
\boldsymbol{A}_0 = \begin{bmatrix} \boldsymbol{H}_1 & \boldsymbol{H}_2 & \ldots & \boldsymbol{H}_{N_B} \\ \boldsymbol{H}_2^T & \boldsymbol{H}_1 & \ldots & \boldsymbol{H}_{N_B-1} \\ \ldots & \ldots & \ldots & \ldots \\ \boldsymbol{H}_{N_B}^T & \boldsymbol{H}_{N_B-1}^T & \ldots & \boldsymbol{H}_1 \end{bmatrix} \tag{6}
$$

where

$$
\boldsymbol{H}_j = \frac{1}{N_B} \sum_{k=-N_B/2}^{N_B/2} c_k e^{-i\alpha k(j-1)} \boldsymbol{A}_S^{(k)} \tag{7}
$$

and N_B is a number of blades in the assembly; $\alpha = 2\pi / N_B$; $c_k = 1$ for all *k*, except $c_k = 0.5$ for $k = N_B/2$ when N_B is even. $\boldsymbol{A}_S^{(k)}$ are so-called 'wave' FRF matrices corresponding to forward travelling (wave number +*k*) and backward travelling (wave number -*k*) waves of engine order excitation. They can be can be expressed through the complex natural modes of one sector only as:

forward travelling wave

$$
\boldsymbol{A}_S^{(k)}(\omega) = \sum_{r=1}^{m} \frac{\phi_r^{(k)} \left(\overline{\phi}_r^{(k)}\right)^T}{(1+i\eta_r)\left(\omega_r^{(k)}\right)^2 - \omega^2}; \tag{8}
$$

back travelling wave

$$
\boldsymbol{A}_S^{(-k)}(\omega) = \sum_{r=1}^{m} \frac{\phi_r^{(k)} \left(\overline{\phi}_r^{(k)}\right)^T}{(1+i\eta_r)\left(\omega_r^{(k)}\right)^2 - \omega^2} \tag{9}
$$

where *m* is a number of modes used for the FRF matrix generation; $\omega_r^{(k)}$ is a natural frequency for the *r*-th mode and *k*-th wave (or cyclic index) number; η_r is the damping loss factor; $\phi_r^{(k)} = \phi_r^{(k)\mathrm{Re}} + i\phi_r^{(k)\mathrm{Im}}$ are mass normalised mode shapes, which are complex for all *k* except $k = 0$ and $N_B/2$; and a line over a symbol denotes complex conjugation. It should be noted that the matrices $\boldsymbol{A}_S^{(k)}$ and $\boldsymbol{A}_S^{(-k)}$ are not Hermitian conjugates due to the presence of damping in the considered system.

It should be noted that Eq.(5) is an exact expression and one of its very useful properties is the possibility it offers of obtaining the response of a mistuned system by considering only a small subset of the degrees of freedom. These degrees of freedom are: (i) those where the mistuning is applied and (ii) those where the forced response levels are of interest, and this combination is referred to below as the **active co-ordinates.**

The distribution of forces over the blade nodes are taken into account when forced response the **tuned** assembly is calculated, which can be done efficiently even for the case when the loads are applied to large number of nodes. Owing to accounting for the forces at the stage of tuned system calculation, the nodes where the forces are applied may not be included into the set the active nodes. An example of active nodes (shown by red circles) and nodes where forces are applied (shown by green circles) is given in Fig.2b.

As a result of these properties, the size of the matrices can be reduced to any desired level without any loss of exactness of the description of the behaviour. This can be achieved as follows. First, the vector of complex displacement amplitudes is partitioned into two: the part with the **active** co-ordinates (index *a*) and the part containing all the other, **passive,** degrees of freedom (index *p*), i.e. $\boldsymbol{q} = \{\boldsymbol{q}_a, \boldsymbol{q}_p\}^T$. The modification, or mistuning, matrix, $\Delta\boldsymbol{Z}$, is also partitioned accordingly, as illustrated below.

$$
\Delta\boldsymbol{Z} = \begin{bmatrix} \Delta\boldsymbol{z}_{aa} & \Delta\boldsymbol{z}_{ap} \\ \Delta\boldsymbol{z}_{pa} & \Delta\boldsymbol{z}_{pp} \end{bmatrix} = \begin{bmatrix} \boldsymbol{U}_a \boldsymbol{V}_a^T & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0} \end{bmatrix} = \begin{bmatrix} \boldsymbol{U}_a \\ \boldsymbol{0} \end{bmatrix} \begin{bmatrix} \boldsymbol{V}_a \\ \boldsymbol{0} \end{bmatrix}^T \tag{10}
$$

Then, by also partitioning the receptance matrix for the tuned system $\boldsymbol{A}_0$ in the same way, the expression in Eq.(5) can be written in the following form:

$$
\begin{Bmatrix} \boldsymbol{q}_a \\ \boldsymbol{q}_p \end{Bmatrix} = \begin{Bmatrix} \boldsymbol{q}_a \\ \boldsymbol{q}_p \end{Bmatrix}_0 - \begin{bmatrix} \boldsymbol{A}_{aa}^0 & \boldsymbol{A}_{ap}^0 \\ \boldsymbol{A}_{pa}^0 & \boldsymbol{A}_{pp}^0 \end{bmatrix} \begin{bmatrix} \boldsymbol{U}_a (\boldsymbol{I} + \boldsymbol{V}_a^T \boldsymbol{A}_{aa}^0 \boldsymbol{U}_a)^{-1} \boldsymbol{V}_a^T & 0 \\ 0 & 0 \end{bmatrix} \begin{Bmatrix} \boldsymbol{q}_a \\ \boldsymbol{q}_p \end{Bmatrix}_0
$$

$$
= \begin{Bmatrix} \boldsymbol{q}_a \\ \boldsymbol{q}_p \end{Bmatrix}_0 - \begin{bmatrix} \boldsymbol{A}_{aa}^0 \boldsymbol{U}_a (\boldsymbol{I} + \boldsymbol{V}_a^T \boldsymbol{A}_{aa}^0 \boldsymbol{U}_a)^{-1} \boldsymbol{V}_a^T \\ \boldsymbol{A}_{pa}^0 \boldsymbol{U}_a (\boldsymbol{I} + \boldsymbol{V}_a^T \boldsymbol{A}_{aa}^0 \boldsymbol{U}_a)^{-1} \boldsymbol{V}_a^T \end{bmatrix} \{\boldsymbol{q}_a\}_0 \tag{11}
$$

where $\boldsymbol{A}_{aa}^0$, $\boldsymbol{A}_{ap}^0$, $\boldsymbol{A}_{pa}^0$, $\boldsymbol{A}_{pp}^0$ are partitions of the matrix, $\boldsymbol{A}_0$, according to the unmodified and modified co-ordinates. An important feature of Eq.(11) is that it expresses the response of the mistuned system as that of the tuned system minus an expression which depends on (i) the FRF matrix of the tuned system, (ii) the mistuning matrix and (iii) the response of the tuned system at the active co-ordinates. As a result, this feature allows calculation of the mistuned response levels at the active co-ordinates by considering only those rows in Eq.(11) which correspond to the active coordinates. The formulation in the next section deals with active coordinates only, hence the index *a* is dropped in subsequent development of the formulation.

2.2 Recurrence update of the forced response

It is seen that Eq.(11) needs the inversion of a square matrix of order *n* which is the number of co-ordinates involved in the mistuning modifications. This matrix inversion process can also be avoided by decomposing the modification matrix into a sum of individual modifications, $\Delta\boldsymbol{Z}=\sum_{j=1}^{n}\Delta\boldsymbol{z}_j$, such that each can be written as a multiplication of two vectors, i.e. $\Delta z_j=\boldsymbol{u}_j\boldsymbol{v}_j^T$. This yields a much simpler formula for calculating the forced response levels of mistuned systems. If the same scheme of matrix decomposition as was illustrated in Fig.2 is adopted here, then $\boldsymbol{u}$ will be a unit vector with one non-zero component and Eq.(5) will be reduced to a simple expression of the form:

$$\boldsymbol{q}=\boldsymbol{q}_0-\frac{\boldsymbol{v}_j^T\boldsymbol{q}_0}{1+\boldsymbol{v}_j^T\boldsymbol{A}_j}\boldsymbol{A}_j \qquad (12)$$

where $\boldsymbol{A}_j$ is the ***j***-th column of matrix $\boldsymbol{A}_0$, and the coefficient before $\boldsymbol{A}_j$ in Eq.(12) is a scalar value. As a result, it is possible to take into account individual rows of the mistuning matrix sequentially, using the following recurrence formula:

$$\boldsymbol{q}^{(j+1)}=\boldsymbol{q}^{(j)}-\frac{\boldsymbol{v}_j^T\boldsymbol{q}^{(j)}}{1+\boldsymbol{v}_j^T\boldsymbol{A}_j^{(j)}}\boldsymbol{A}_j^{(j)} \qquad (13)$$

$$\boldsymbol{A}^{(j+1)}=\boldsymbol{A}^{(j)}-\frac{\boldsymbol{A}_j^{(j)}\left(\boldsymbol{v}_j^T\boldsymbol{A}^{(j)}\right)}{1+\boldsymbol{v}_j^T\boldsymbol{A}_j^{(j)}} \qquad (14)$$

To start the recurrence update, the response amplitudes and the FRF matrix of the tuned bladed disc are used.

3. Mistuning modelling

3.1 Mistuning elements

The method for the analysis for mistuned systems proposed does not make any assumptions regarding the distribution or the magnitude of the mistuning matrix, $\Delta\boldsymbol{Z}$. However, it is useful to define, without any loss of generality, some simple mistuning elements that can be applied at desired locations in the bladed disc model and have physical interpretation. Examples of such elements are lumped mass, stiffness and damping mistuning and the mistuning element matrix, $\Delta\boldsymbol{z}^e$, can be expressed as:

mass mistuning

$$\Delta\boldsymbol{z}^e=\begin{bmatrix}-\omega^2 m_x & 0 & 0\\ 0 & -\omega^2 m_y & 0\\ 0 & 0 & -\omega^2 m_z\end{bmatrix} \qquad (15)$$

stiffness mistuning   damping mistuning

$$\Delta\boldsymbol{z}^e=\begin{bmatrix}k_x & 0 & 0\\ 0 & k_y & 0\\ 0 & 0 & k_z\end{bmatrix};\qquad \Delta\boldsymbol{z}^e=\begin{bmatrix}id_x & 0 & 0\\ 0 & id_y & 0\\ 0 & 0 & id_z\end{bmatrix} \qquad (6)$$

where m_j, k_j; d_j are mass, stiffness and damping coefficients of the element in three orthogonal directions. Their combination allows us to describe a wide range of mistuning patterns that might be encountered in practice.

3.2 Blade frequency mistuning

It is possible to describe quite a wide class of possible mistuning arrangements using the simple mistuning elements introduced in the previous section. One of the widely-used measures of mistuning in practice is to refer to the 'blade-alone' cantilever-natural-frequency scatter. This type of mistuning can easily be represented by establishing a relationship between individual blade frequencies and the properties of a set of mistuning elements applied to the blades. This can be achieved by conducting a preliminary analysis to determine a one-to-one relationship between the blade alone frequency scatter and the amount of mistuning introduced at specific locations. A possible approach is to express the mistuning matrix for a blade, $\Delta\boldsymbol{Z}$, as

$$\Delta\boldsymbol{Z}=c_f\Delta\boldsymbol{Z}_0 \qquad (17)$$

where $\Delta\boldsymbol{Z}_0$ is a predefined mistuning matrix for a given set of mistuning element characteristics and c_f is a variable representing mistuning coefficient. Calculations are made for a sufficient number of values of c_f so as to establish a relationship with an acceptable accuracy between the mistuning coefficient, c_f, and the scatter in blade-alone frequency. This relationship is usually non-linear but a very good description can be made using a spline approximation. An example demonstrating such relationship is shown in Fig.3 for the first blade-alone natural frequency when lumped mass mistuning elements are applied to active nodes shown in Fig.2b.

It should be stated that some preliminary blade-alone analysis may be necessary to determine the datum mistuning matrix, $\Delta\boldsymbol{Z}_0$, especially if the blade-alone frequency mistuning for more than one mode of vibration is of interest.

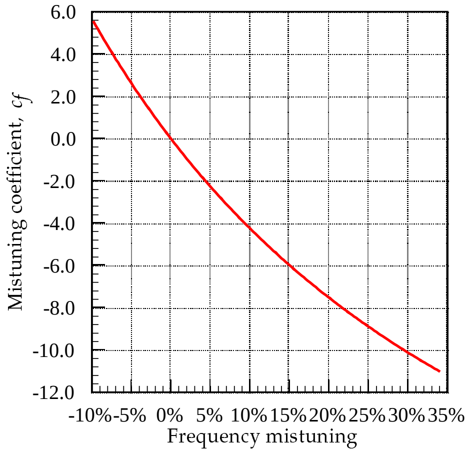


Fig. 3 Relationship between mistuning coefficient and frequency mistuning

4. Numerical results

4.1 The bladed disc analysed and excitation conditions

The method developed above has been implemented in FORTRAN program and has been applied to the response analysis of a number of realistic mistuned bladed discs.

In the present section, an example of the method applied to the analysis of a high-pressure turbine bladed disc shown in Fig.1 is represented. The bladed disc analysed comprises 92 blades. Its finite element single-sector model shown in Fig.2a contains 162,708 degrees of freedom (DOFs) and the full bladed disc comprises about 15 millions DOFs. The model describes, in detail, the geometric shape of the blade and the disc, the blade-disc interface and the anisotropy of the blade material. The sector model is used for determination the first 16 natural frequencies and mode shapes of the tuned bladed-disc assembly for all their cyclic indices (nodal diameters) that are possible for the considered assembly in the range from 0 to 46. These modes of the tuned system are used for generating the FRF matrix of the tuned assembly.

A relatively small subset of nodes was selected for analysis of the mistuned system. The nodes used are shown in Fig.2b, where green circles indicate the nodes subjected to loads and red circles are the active nodes where response amplitudes are calculated and condensed mass mistuning elements are applied. The blade mistuning is assumed random and is normally distributed in the blade set manufactured. Realistic characteristics of the normal distributions (i.e. the mean value and the standard deviation) for first four blade-alone natural frequencies (namely, first flap, 1F, and first torsional, 1T) are determined as a result of statistical analysis of several hundreds of the experimental values. The mistuning patterns analysed are then generated using a random number generator for the obtained distribution characteristics.

Moreover, the influence on response levels of damaged blades, which can have large mistuning levels, is analysed. In the cases analysed here, the damaged blades can reach 34% discrepancy in the first natural frequency compared with the corresponding value for the tuned blade. In the mistuning patterns analysed, damaged blade mistuning values replace mistuning values for blades of the randomly-generated mistuning pattern. The study of the influence of a small number of blades with very different frequencies was undertaken to assess the following aspects:

- Assess the damage tolerance of the assembly, assuming a damaged blade would have different dynamic characteristics to standard blades.
- What effect would introducing blades that may be outside normal acceptance limits have and could they be used without adverse effects? If the effect was relatively benign then it may be possible to extend acceptance limits and therefore reduce costs.

Excitation by 6^{th}, 7^{th} and 8^{th} engine orders (EO) was studied in the frequency range corresponding to the 1F mode, and excitation by 40^{th} EO was studied in the frequency ranges corresponding to the 1T mode. Loading for each blade was distributed uniformly over the blade nodes and the damping loss factor was assumed equal 0.003. Excitation frequency ranges for each of EOs analysed, natural frequencies of the tuned assembly and mean values for experimental natural frequencies are shown in Fig.4. The frequency values were normalised with respect to the first blade-alone natural frequency, i.e. actual values were divided by a value of the first blade-alone natural frequency.

4.2 An estimate of mistuning effect on response level and amplitude distribution

An example of the calculated amplitudes as functions of excitation frequency for all blades of the bladed disc is shown in Fig.5 for the case of excitation by 6EO. Fig.5a shows the blade amplitudes for the case of a tuned bladed disc; Fig.5b illustrates the blade amplitudes for randomly mistuned bladed disc within the normal mistuning

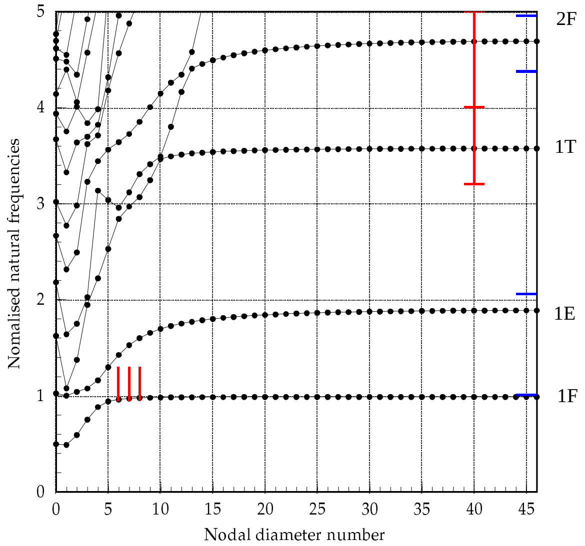
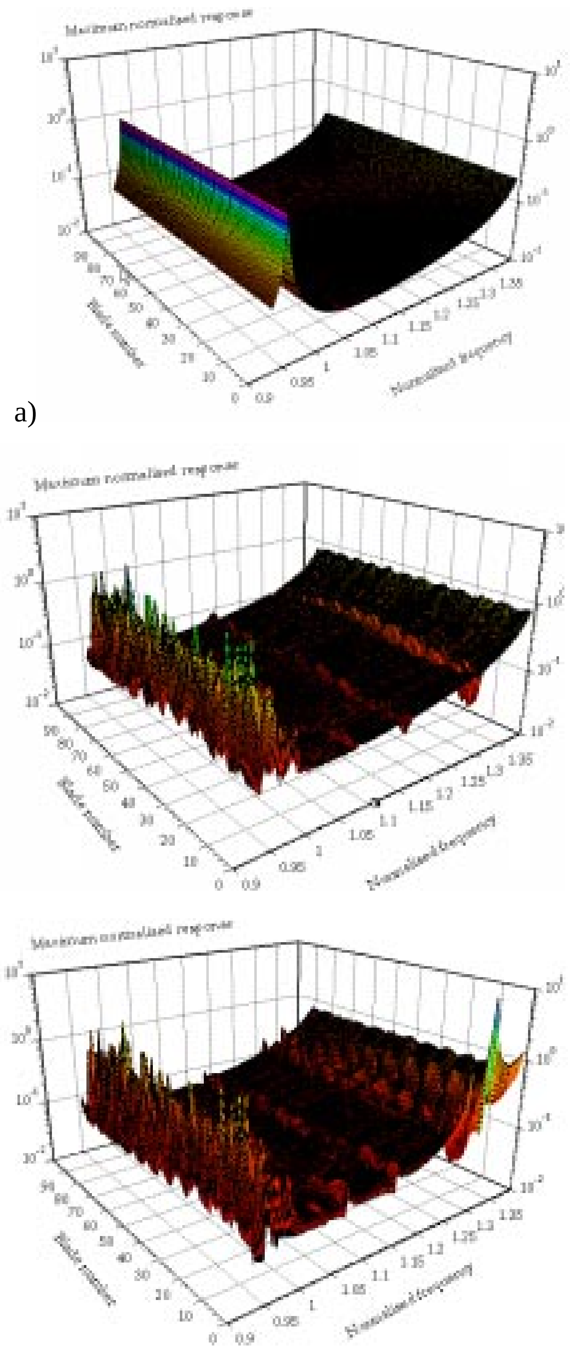


Fig. 4 Natural frequency of the tuned assembly and analysed excitation engine orders, excitation frequency ranges (red lines) and mean values for experimental natural frequencies (blue lines)



a)

b)

c)

Fig. 5 Forced response for each blade of the bladed disc: a) tuned assembly; b) randomly mistuned blades; c) random mistuning and a damaged blade. The case of 6EO excitation in 1F mode frequency range.

range; and Fig.5c contains the blade amplitudes of the bladed disc having 90 randomly mistuned blades plus two damaged blades (namely, the first and fifteenth blades). All amplitudes are normalised by a value of the maximum amplitude of the tuned assembly. As one can see, the tuned bladed disc has the same amplitudes for all blades and there is only one resonance peak in the range considered – that is the 6th nodal diameter mode. The introduction of even a small amount of mistuning leads to significant amplitude scatter over the blades of the bladed disc as well as to the appearance of a large number of resonance peaks. The highest peaks of the system with small mistuning lie near the resonance peak of the tuned assembly with the normalised frequency value close to 1. The randomly-mistuned bladed disc with the damaged blades has two additional resonance peaks in the vicinity of normalised frequency magnitude 1.35. Resonance frequencies corresponding to these peaks are close to the natural frequencies of the blade-alone damaged blades.

Envelopes of the maximum responses found over all blades of the assembly are shown in Fig.6 for all the considered excitation types. Here, maximum normalised amplitudes are compared for the tuned bladed disc, for the randomly-mistuned assembly with a normal, small, mistuning level, and for the randomly-mistuned assembly having two damaged blades. For the case of 6EO excitation in the range of the 1F mode, one can see that small random mistuning gives rise to many resonance peaks. These peaks lie in the vicinity of the single resonance peak of the tuned system and far from it as well.

The highest resonance peaks of the mistuned system are close to the resonance of the tuned assembly. Some of these peaks are higher than the resonance peak of the tuned system although many of them are lower. Owing to the presence of many resonances, the envelope of the maximum response has a much wider frequency range with high amplitudes than the one forthe tuned system. For

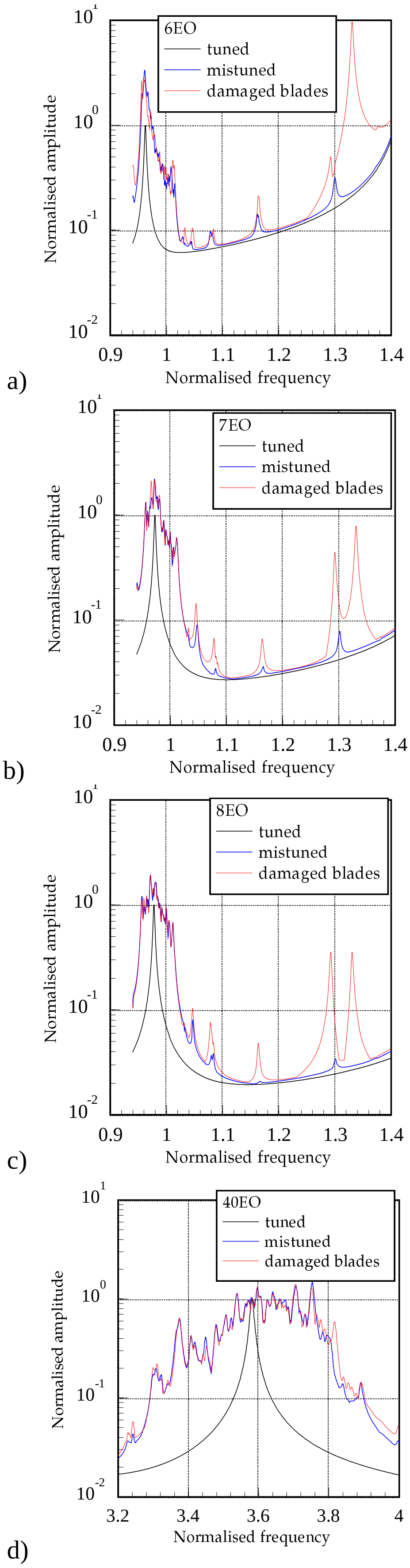


Fig. 6 Comparison of envelopes of maximum normalised response for tuned assembly, randomly mistuned one and the bladed disc containing damaged blades.

the case of the bladed disc with the damaged blades, two new resonance peaks appear near the natural frequencies of the damaged blades and also near other assembly modes in the mistuned system. For the case of excitation by 6EO maximum normalised amplitude of such bladed disc has a value which is much higher than worsening effect caused by normal mistuning. The maximum response for the cases of excitation by 7EO and 8EO also lead to similar conclusions about the influence of small mistuning and damaged blades to forced response. The only difference is the much lower levels of normalised amplitude (less than 1) for the resonance peaks in the vicinity of the natural frequencies of the damaged blades. Analysing excitation by 40EO in the 1T frequency range in Fig.6d, one can see that the introduction of small random mistuning in this case leads to a large increase in the frequency band with large amplitudes compared with that of the tuned system. This frequency range increase is essentially greater than for the case of excitation in the 1F mode frequency range. As is typical for mistuned assemblies, the maximum resonance peaks are higher than those of the tuned bladed disc. The damaged blades for these cases do not change very much maximum response level and character of the envelope of forced response.

Correspondence between distinctive resonance peaks of the randomly-mistuned bladed disc and natural frequencies of its tuned counterpart is demonstrated in Fig.7. Here, six distinctive resonance peaks are numbered in Fig.7a showing the envelope of the response, and the natural frequencies of the tuned system corresponding them are shown in Fig.7b. All these peaks are on sloping parts of curves in the natural frequencies-nodal diameters diagram, where blade vibration coupling through the disc is high. Many resonance peaks corresponding to the natural frequencies lying on the flat parts of the first curve form a relatively wide normalised frequency range (from 0.96 to 1.0) with high response levels. This range is formed by combined resonance peaks, which are almost indistinguishable.

In Fig.8 distributions of amplitudes over all the blades of the bladed disc are displayed which correspond to the first four of the distinctive resonance peaks shown in Fig.7. One can see that one or a small number of blades can have much higher amplitudes than the others for some resonances. For the 4th resonance peak analysed, the distributions of amplitudes have a regular pattern. This peak corresponds to the natural frequency of the tuned system, which is distant from any others.

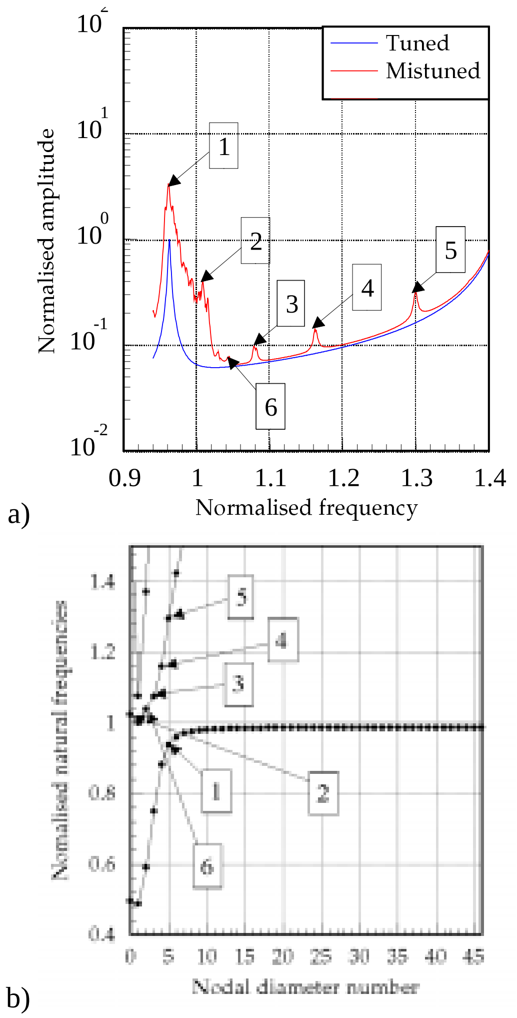


Fig. 7 Correspondence of the distinctive resonance peaks for the mistuned assembly a) and natural frequencies (b) of the tuned one: 6EO excitation

4.3 *Forced response analysis for randomly-generated mistuning patterns*

A statistical survey of the worsening effect of mistuning has been carried out for many randomly generated patterns. A small reduction of the frequency ranges used for the case of statistical calculations was made to compare maximum amplitudes of the required modes (1F/6EO, 1F/7EO, 1F/8EO, and 1T/40EO) only. Such reductions in the frequency range were made after performing an FRF analysis, which showed high amplitudes near boundaries of the considered frequency ranges. These amplitudes were determined by interactions of a damaged blade vibrating in one mode and the bladed disc assembly vibrating in another family of modes. The latter case appeared for the bladed discs with damaged blades owing to their large frequency deviations from the mean value for the whole blade population.

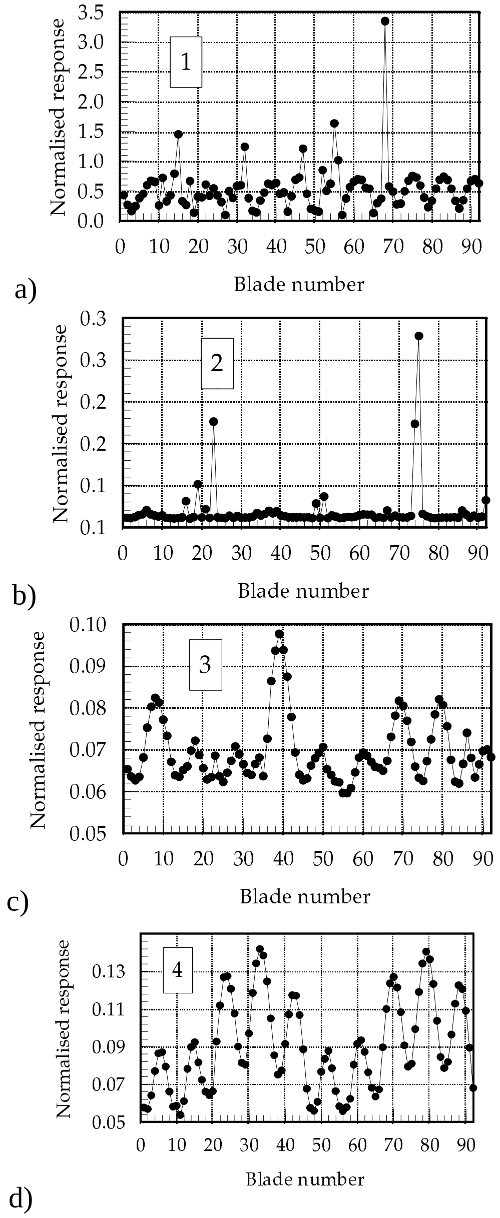
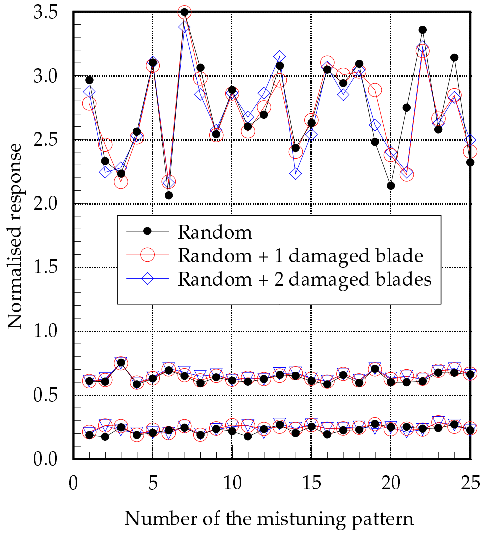
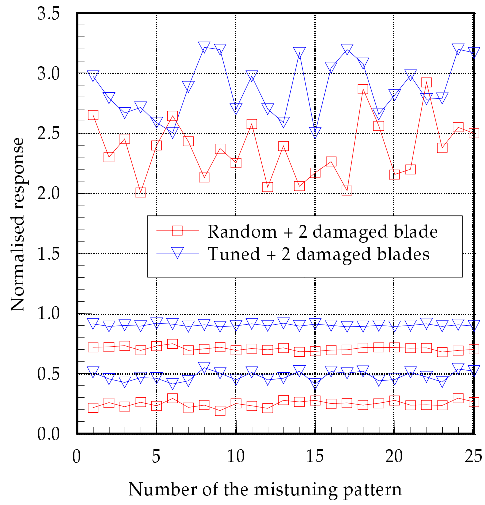


Fig. 8 Amplitude distribution over the bladed disc at the distinctive resonances

The maximum, mean and minimum amplitudes for each of the analysed mistuning patterns are shown in Figs.9-10. The maximum amplitude was found over all blades of the mistuned bladed discs and among all excitation frequencies from the range considered. The mean amplitude was determined as the sum of the maximum amplitudes found for each blade in the considered excitation frequency range divided by the number of blades. The minimum amplitude was determined as the value minimum found from the maximum amplitudes found for each blade in the considered range.

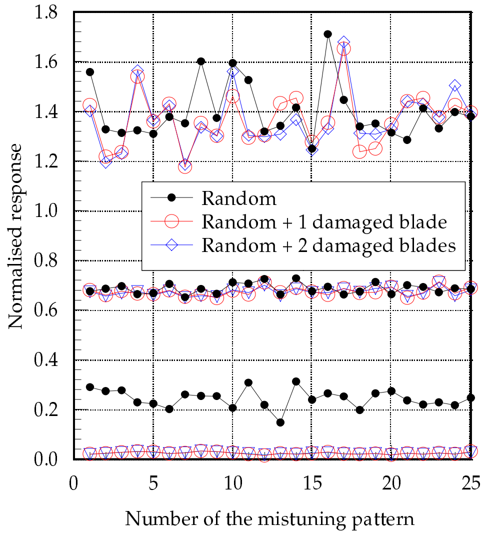


a)

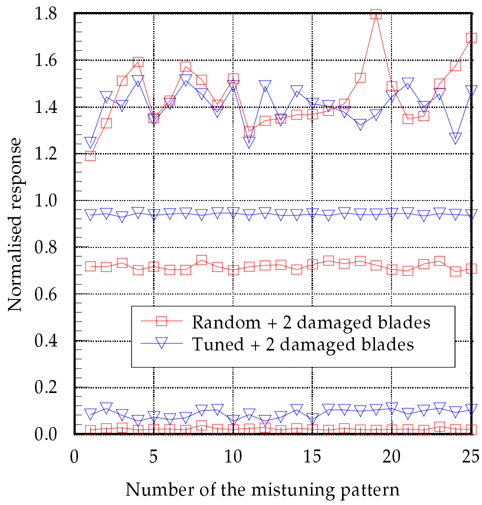


b)

Fig. 9 The case of 6EO excitation in normalised frequency range 0.94…1.4. Maximum, mean and minimum amplitudes found over all blades for each mistuning patterns analysed: a) randomly generated mistuning patterns; b) randomly reshuffled blades from a given set



a)



b)

Fig. 10 The case of 40EO excitation in normalised frequency range 3.2…4. Maximum, mean and minimum amplitudes found over all blades for each mistuning patterns analysed: a) randomly generated mistuning patterns; b) randomly reshuffled blades from a given set

In the first plot (a) of each of Figs.9-10, the amplitudes are given for the following types of mistuning:

1) random blade mistuning within normal mistuning range (M1);
2) one damaged blade and the rest blades are randomly mistuned (M2);
3) two damaged blades and the rest blades are randomly mistuned (M3).

In total, 25 different mistuning patterns were generated for each of these types. For the 2nd and 3rd types the positions of the damaged blades were fixed while for the blades with normal mistuning range, the actual frequency mistune pattern was generated using characteristics of the fitted normal distribution.

For the second plot in each of these figures, (b), the following types of mistuning are considered:

1) two damaged blades and the rest are tuned, positions of the damaged blades are randomly changed (M4);
2) 90 experimentally-determined mistuned blades with 2 damaged blades which are reshuffled changing positions of the all 92 mistuned blades in the bladed disc (M5).

For this plot random reshuffling of the initial mistuning pattern was performed (i.e. all 25 cases used the same set of blades, but rearranged in different order). To facilitate references the five mistuning types considered are marked here by symbols M1…M5.

For example one can see that for the 1F mode excited by 6EO (Fig.9) the introduction of the damaged blades into a randomly-mistuned assembly results in only small changes of maximum, mean and minimum amplitudes. The effect of damaged blades on maximum response levels is more significant here for the case of blade reshuffling.

In Table 1 some statistical characteristics for all the analysed mistuning patterns are given. Namely, the mean values and scatter ranges over all the analysed mistuning patterns are determined for the above-mentioned maximum, mean and minimum amplitudes found for each mistuning pattern. One can see that the scatter of maximum amplitudes usually lies within the range ±30% of the average and for mistuning patterns obtained by blade reshuffling this scatter is smaller than for the case of randomly-generated mistuned blade sets. The scatter is especially small for the case when only two blades are mistuned and their positions are changed as a result of random reshuffling. The scatter of mean amplitudes is small and is usually measured in a few percent.

Table 1 Mean values and scatter of the normalised response for all analysed mistuning patterns: 6EO excitation

Type of mistuning (see page 10)	Maximum normalised response		Mean normalised response	
	Mean for all patterns	Scatter, %	Mean for all patterns	Scatter, %
M1	2.72	-20…28	0.65	-8…15
M2	2.74	-25…28	0.63	-8…19
M3	2.71	-20…25	0.65	-8…15
M4	2.87	-13…12	0.90	-1.3…1.9
M5	2.37	-15…23	0.70	-4.0…5.9

The maximum normalised amplitudes found over all the analysed mistuning patterns are given for each kind of excitation Table 2. The worsening effect of the mistuning is largest for the 1F/6EO resonance condition for the considered bladed disc and excitation conditions. It results in more than three times higher amplitudes for some blades than for the tuned system. The least mistuning worsening effect is for 1F/8EO and 1T/40EO excitations, when the maximum normalised amplitudes are around 2.3 and 1.8 respectively. Introduction of the damaged blades does not affect the worsening much when frequency ranges including resonances determined by the required blade mode only (i.e. by 1F or 1T modes) are considered. However, it should be noticed (see Fig.6) that the damaged blades caused the appearance of large resonance peaks in the widened frequency ranges, where the damaged blade natural frequencies are close to natural frequencies of the bladed-disc assembly.

Table 2 Maximum normalised amplitude levels found over the all analysed mistuning patterns

Type of mistuning (see page 10)	Kind of excitation			
	1F/6EO	1F/7EO	1F/8EO	1T/40EO
M1	3.495	2.786	2.290	1.709
M2	3.493	2.773	2.163	1.651
M3	3.379	2.756	2.009	1.679
M4	3.211	2.710	1.457	1.512
M5	2.921	2.676	2.103	1.795
Total over all the types	3.495	2.786	2.290	1.795

5. Conclusions

An efficient method for the forced vibration response analysis of mistuned bladed discs using detailed finite element models has been presented. The method proposed here allows the exact calculation of the response of mistuned bladed discs using FRF matrices of the corresponding tuned system and together with a mistuning matrix. The distinct features of the proposed method are: (i) only a single sector model is needed to represent the tuned and mistuned bladed disc, (ii) mistuning is treated as a structural modification problem, (iii) the computational cost for mistuning calculations does not depend on the size of the original sector model as the solution is obtained at active co-ordinates, and (iv) the reduced model corresponding to the active co-ordinates is as accurate as the initial sector model represented by its natural frequencies and mode shapes.

Investigations of the forced response of mistuned bladed discs containing randomly mistuned blades (within the normal frequency mistuning range), and with damaged blades, have been carried out. The worsening effect of mistuning on resonance amplitude levels and on forced response functions has been analysed using detailed, realistic finite element model. The highest amplitude level was determined from many mistuning patterns. These mistuning patterns were created by: (i) generation frequency mistuning for each individual blade randomly; (ii) random reshuffling a predefined mistuning set.

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