Research Statement

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My research interests are split into two main areas. On the theoretical side, I work on homological problems in noncommutative geometry (NCG). The roots of the subject lie in Hilbert's Nullstellensatz and Gelfand-Naimark Theorem [42]. NCG asserts that the category of algebras (both commutative and noncommutative alike) *de facto* is the category of spaces: while ordinary spaces correspond to commutative algebras, noncommutative algebras too correspond to certain virtual (noncommutative) *spaces*. It strives to build a comprehensive dictionary between geometry and algebra going beyond classical algebraic geometry à la Grothendieck. The bulk of my published research is on extension of the cyclic and Hochschild (co)homology of noncommutative algebras into the Hopf-equivariant direction [73, 79, 80, 74, 81, 76, 75, 77, 78, 84, 83, 69]

On the applied side, my research is firmly grounded in applied statistics within the context of Ottoman history. The Ottoman historiography, while quite mature, is rather short on quantitative studies. In a continuing long term project with my historian colleague Prof. Ergene from University of Vermont, we analyze various Ottoman sources using techniques and methods hitherto only used by our Western historian colleagues [12, 97, 99, 112, 131]. While I processed the raw data and performed all of the statistical and computational work on the processed data by using state-of-the-art machine learning and statistical tools, Prof. Ergene's careful work provided the contextual and historical analyses of the results we obtained. Our collaboration already produced four articles which are firsts in this area [31, 32, 34, 33].

The algorithmic nature of many cohomological problems I work on frequently generate problems in linear algebra, combinatorics and graph theory. Also, my interactions with colleagues working in different disciplines exposed me concrete computational problems which can be reduced to statistical, combinatorial or graph theoretical questions. My typical workflow incorporates computational software such as Sage, Singular, GAP, Octave, R and python. My long-time personal interest in computing, programming languages and algorithms puts me a unique position to have access to both highly abstract and theoretical tools such as category theory and abstract homotopy theory as well as very concrete computational tools such as graph algorithms, statistical tests and their implementations to solve such problem. The tool set I command helped me in the past to solve a wide range of problems such as the thorny practical problems in processing of and analyzing the raw data that my colleague Prof. Ergene collected [31, 32, 34, 33], and resolving a long-standing difficult theoretical problem of deciding whether all Connes-Moscovici pairings defined in the literature [46, 21, 85, 127, 122] were indeed the same using derived categories and bifunctors [75, 77].

Below, I will summarize my past and current research by providing also necessary context in which these results become relevant. I will also provide an abridged version of my future research plans.

1. Research Background

1.1. Mathematics.

1.1.1. *Loday-Quillen-Tsygan theorem for coalgebras.* The space $BGL(A)^+$ which is defined by Quillen [121] to define algebraic K-theory of a rational algebra A, is an H-space whose first fundamental group is abelian. Thus its rational cohomology $H^*(BGL(A)^+, \mathbb{Q})$ is a commutative and cocommutative Hopf algebra. Then by Milnor-Moore [113], the cohomology is an exterior algebra over its submodule of primitive elements which is composed of rational K-groups $K_*(A)_\mathbb{Q}$ [95]. The cohomology of the Lie algebra $\mathfrak{gl}(A)$ with rational coefficients is also a commutative and cocommutative Hopf algebra, and therefore, is generated by its primitive elements. Tsygan [133] and Loday-Quillen [96] proved that the submodule of primitive elements in this case consists of the cyclic cohomology groups $HC^*(A)$. In [72] I proved the coalgebra analogue of the Loday-Quillen-Tsygan Theorem: for a rational coalgebra C, the Lie coalgebra homology of the Lie coalgebra $\mathfrak{gl}(C)$ is a commutative cocommutative Hopf algebra which is generated by its submodule of primitive elements which consists of cyclic homology of the coalgebra $HC_*(C)$.

1.1.2. *Hopf-Hochschild cohomology of (co)module (co)algebras.* There is a rather intriguing gap in the existing literature about the effect of Hopf symmetries on various (co)homology of algebras. To fill this curious gap, in [74] I defined a Hopf-equivariant version of the Hochschild cohomology, and showed that in low dimensions one gets the Hopf-equivariant analogue of the ordinary Hochschild (co)homology classes. In the same paper, I also showed that the theory satisfied Morita invariance. In the light of Hochschild-Kostant-Rosenberg Theorem [57], and the result that Hopf-Hochschild cochain complex admits Hopf-equivariant Gerstenhaber bracket [141], one should view this Hopf-equivariant Hochschild cohomology as the right analogue of G-invariant differential forms of a space admitting an action of a group G.

1.1.3. *Hopf cyclic cohomology of (co)module (co)algebras.* In the article based on my PhD thesis [73], I gave a fundamentally different description of the cohomology of Hopf-algebras defined by Connes and Moscovici [15, 16]. The original definition given by Connes and Moscovici was formulated for a specific class of Hopf algebras and crucially used the invertibility of the antipode of the underlying Hopf algebra. Even though this theory was immediately extended to a wider class of (co)algebras in [54, 53], the extension again depended crucially on the existence of an invertible antipode. The new conceptual definition I gave, by removing the non-trivial dependence on the antipode, widened the scope of the theory to a much larger category of (co)algebras. This Hopf-equivariant cohomology theory, which also admits twisting coefficients, is a variation of the ordinary cyclic cohomology of (co)algebras with Hopf equivariance built in. It is the right noncommutative analogue of the equivariant cohomology of spaces admitting a group action.

1.1.4. *Categorical definition of Hopf-cyclic cohomology.* Immediately after the publication of [73], in a series of articles with M. Khalkhali, we investigated formal properties of this equivariant cohomology of module (co)algebras. We showed in [79] that this equivariant cohomology theory, as well as the theories it extended [15, 16, 53], all satisfy excision. Then in [81], we proved that these cohomology theories satisfies Morita invariance and they can also be described as a homotopy localization of a specific category. Finally, I proved in [76] that such Hopf-equivariant cohomology theories can be defined purely categorically, for a monoid object in an arbitrary symmetric monoidal category enjoying an action of a monad.

1.1.5. *Jacobi-Zariski exact sequence.* An extension $\mathcal{B} \subseteq \mathcal{A}$ of k-algebras are called *reduced-flat* (*r-flat* in short) extensions if the quotient $\mathcal{B}$-bimodule $\mathcal{A}/\mathcal{B}$ is flat. In [78] I showed that for r-flat extensions $\mathcal{B} \subseteq \mathcal{A}$, there are long exact sequences for ordinary (co)homology, Hochschild homology and cyclic (co)homology of k-algebras of the form (written here for Hochschild homology)

$$\cdots \to HH_{n+1}(\mathcal{A}|\mathcal{B}) \to HH_n(\mathcal{B}|k) \to HH_n(\mathcal{A}|k) \to HH_n(\mathcal{A}|\mathcal{B}) \to \cdots$$

There are similar long exact sequences in the literature such as the Wodzicki excision sequence [140] for Hochschild homology and cyclic (co)homology (written here for Hochschild homology)

$$\cdots \to HH_n(\mathcal{I}) \to HH_n(\mathcal{B}) \to HH_n(\mathcal{A}) \to HH_{n-1}(\mathcal{I}) \cdots$$

for an epimorphism $\pi\colon \mathcal{B} \to \mathcal{A}$ of unital k-algebras with an H-unital kernel $\mathcal{I} := ker(\pi)$. The Wodzicki excision sequence characterizes homotopy cofiber of the morphism of differential graded k-modules $\pi_*\colon CH_*(\mathcal{B}) \to CH_*(\mathcal{A})$ induced by π as the suspended Hochschild complex $\Sigma CH_*(\mathcal{I})$ of the ideal $\mathcal{I}$. This new variation of the Jacobi-Zariski sequence characterizes the same homotopy cofiber as the relative Hochschild chain complex $CH_*(\mathcal{A}|\mathcal{B})$ (relative à la Hochschild [56]) for a monomorphism $\mathcal{B} \to \mathcal{A}$ of k-algebras.

Now, assume $\varphi\colon \mathcal{B} \to \mathcal{A}$ is an arbitrary morphism of unital k-algebras such that $\mathcal{I} := ker(\varphi)$ is H-unital and the quotient $\mathcal{B}$-module $\mathcal{A}/im(\varphi)$ is flat. Under these conditions, I showed (written here for Hochschild chain complexes) that homotopy cofiber $CH_*(\mathcal{A}, \mathcal{B})$ of the morphism $\varphi_*\colon CH_*(\mathcal{B}) \to CH_*(\mathcal{A})$ induced by φ fits into a homotopy cofibration sequence of the form

$$\Sigma CH_*(\mathcal{I}) \to CH_*(\mathcal{A}, \mathcal{B}) \to CH_*(\mathcal{A}|\mathcal{B})$$

which yields an appropriate long exact sequence unites Wodzicki's characterization of the homotopy cofiber when φ is an epimorphism, and our Jacobi-Zariski characterization when φ is a monomorphism.

1.1.6. *Global and finitistic dimension problems for Artinian algebras.* Artinian algebras whose lattice of ideals enjoy convenient finiteness properties are attractive objects precisely because of this finiteness condition. Moreover, they can be described and investigated through certain combinatorial and graph theoretical devices such as Auslander-Reiten quivers [5] and the natural quiver of an Artinian algebra [90]. In my on-going collaboration with Prof. Kanuni from Düzce University, we have been working on calculating global dimensions of a class of Artinian algebras in terms of these combinatorial objects and certain subalgebras of Artinian algebras. Our preliminary results are promising [68] and indicate that one can attack Bass' Finitistic Dimension Conjectures [6] for Artinian algebras from this perspective.

1.1.7. *Quantum groups and their various Hopf-equivariant cohomologies.* Quantum group algebras and their enveloping algebras have been studied well from the perspective of representation theory [105, 70, 86]. However, the existing literature in the direction of Hochschild and Cyclic cohomology of quantum groups are meek due to combinatorial difficulties in constructing small resolutions for modules over these algebras [108, 109, 21, 48, 50, 49, 47].

In our ongoing project of calculating Hopf-cyclic cohomology of various types of Hopf algebras and their (co)module (co)algebras, Prof. Sütlü from Isik University, we calculated the Hopf-cyclic cohomology of all quantum enveloping algebras $U_q(\mathfrak{g})$ in [84]. As expected, the cohomology is determined by the rank, i.e. the topological dimension of the maximal torus, of the underlying Lie algebra. However, the number of classes appearing indicated that ordinary Hopf-cyclic cohomology of such quantum groups did ignore different orientations one can choose for these tori. In [82] we showed that the dihedral variant of cyclic cohomology [93, 18, 19], which we called Hopf-dihedral cohomology, detects different orientation choices, and up to these choices, now Hopf-dihedral cohomology of quantum groups do indeed have the right number of classes and at the right degree.

1.1.8. *Connes-Moscovici characteristic map and its extensions.* As in any equivariant cohomology, there is a cup product in Hopf cyclic cohomology. However, there were a multitude of such constructions in the literature with no apparent connection between them [46, 21, 85, 127, 122]. In a series of two papers [75, 77] I showed that there is a universal product map in Hopf cyclic cohomology which agrees with the characteristic map of Connes and Moscovici [16], and that all such products in the literature are naturally isomorphic as natural transformation of double functors.

In our calculations Sütlü and myself successfully made for the cohomologies of quantum enveloping algebras [84, 82], we observed that the calculation for quantum group algebras $\mathcal{O}(G_q)$ remained difficult with the current tools at hand. One new way to approach this problem is to find an analogue of a van Est type map $HC^*(U_q(\mathfrak{g})) \to HC^*(\mathcal{O}(G_q))$ [136, 137, 138]. This strategy worked well for Connes and Moscovici for finding characteristic classes of codimension-n foliations using the Hopf-cyclic cohomology of the Hopf algebra $\mathcal{H}_n$ [15, 17]. Since we know by [75, 77] that all characteristic maps do yield the same results, Connes-Moscovici characteristic map seemed to be a solid path to approach this problem. In [83] Sütlü and I were able to construct such a map and showed that some of the classes calculated in the literature [108, 109, 21, 48, 50, 49, 47] are indeed in the image of this characteristic map.

1.1.9. *Coalgebraic methods in cohomology calculations.* There is a nice duality between algebras and coalgebras that work only in one direction: while the vector space dual of a coalgebra is an algebra, the reverse need not be true. However, there is a nice topology on any infinite dimensional vector space and one can extend the duality for algebras if one takes *continuous* dual of the underlying vector space. Abrams and Weibel proved in [1] that the continuous Hochschild cohomology of the algebra $C^\vee$ of a coalgebra is isomorphic to the coalgebra cohomology of C. Kanuni, Sütlü and myself were able to calculate the continuous Hochschild cohomology of a large class of incidence algebras in [69] using Weibel-Abrams result along with the tools we developed calculating coalgebraic cohomology of incidence coalgebras and Hopf algebras.

1.2. **Applied statistics.**

1.2.1. *Social mobility patterns on the periphery of the Ottoman Empire.* The subject of intergenerational mobility has been well studied from a variety of disciplinary perspectives. There is by now an extensive literature, especially for North American and European societies, that examines children's relative tendencies to inherit their parents' wealth/income levels, social positions, occupational identities, and class characteristics as indicators of the potential for cross-generational socioeconomic advancement. [12, 97, 98, 65, 112, 131, 114, 99] The general consensus in these studies is that an individual's socioeconomic status is based more on his/her personal talents, skills, and accomplishments, and less on the circumstances into which s/he was born when intergenerational mobility is high. Economic and sociological analyses on social and economic mobility investigate relationships between individuals' relative access to education and socioeconomic advancement across generations, the roles that labor markets play in structuring job opportunities, how political institutions influence long-term socioeconomic fluidity, the links between marital and intergenerational mobility, and how gender affects access to socioeconomic advancement. In our joint work [31] with Prof. Ergene which marked the beginning our collaboration, we set out to apply sophisticated statistical methods hitherto used to analyze the Western historical milieus, to study intergenerational mobility patterns in eighteenth-century Kastamonu in Ottoman Anatolia. Unlike their Western counterparts, our sources in general lack detailed genealogical data which forced us to make use of proxy data based on probate estate inventories found in court registers. This required a careful processing of raw data to make it suitable for our analyses after which we measured, using state-of-the-art statistical analyses of categorical data [2] of the sort we had, how socioeconomic characteristics were passed from fathers to sons and then we analyzed our findings in relation to structural inequalities and the long-term continuity of class identities.

1.2.2. *Partner choice in 18th century Ottoman Empire.* Marital associations reveal considerable amount of information on the social, cultural, and economic characteristics of a community, which is why the topic of partner-choice has been studied extensively within numerous West European and Northern American societies. These are too numerous to cite in full, but we make note of the following [24, 25, 66, 67, 100, 101]. Collectively, the large body of research demonstrates how spousal selection facilitates reproduction and transmission of socioeconomic status through marriage arrangements. While the historical and sociological literature provides ample information on variations in partner-choice patterns in many Western societies, scholarly research provide little or no information on spousal preferences in non-Western contexts, historical or modern. In Ottoman history-writing only a very small number studies on spousal selection exists to date. In our continuing collaboration with Prof. Ergene, in [32] we investigated marital preferences of different social strata using a different set of variables on the same dataset we constructed in our earlier work [31]. What separates our work from the small number of previous works is the quantitative methodology that we used to produce a nuanced analysis of the dynamics of spousal selections. Our methodology, which is analogous to methods used in partner-choice and stratification research in the Western contexts, allowed us to produce statistically robust answers to many questions raised by the earlier studies in the Western context, and also helped us to pursue many relevant issues specific to the Ottoman context that have not been examined prior to our study. We also aimed to provide other researchers with a set of historical techniques and approaches that are suitable for a source base which is widely available in other Ottoman contexts.

1.2.3. *Wealth and Inequality in Ottoman Empire.* Continuing our collaboration with Prof. Ergene, this time working with an economic historian Prof. Cosgel from University of Connecticut, we investigated how the wealth levels and inequality rates changed in Ottoman Empire using post-mortem estate inventories found in court records [34].

Some economic historians believe that modern economic growth in Europe predated industrial revolution due to advances in agricultural technologies aided by commercial developments which later generated a robust economic growth in the early modern age, and induced rapid industrialization. One can test the validity of this hypothesis by observing differences between the regions that would later be industrialized versus those that would not. First, we tested the validity of this hypothesis by examining the temporal variations in the wealth levels of a representative Middle East context. Second, we provided the first quantitative observations on the relationship between wealth and inequality trends in the same context. This is important because according to Kuznets [88], and later Van Zanden [139], there is a quantifiable relationship between economic development and inequality in the pre-industrial Europe based on the observation that West European economic development between the late-eighteenth and late-nineteenth centuries was concurrent with an increasing level of inequality which disappeared by the end of the nineteenth century. To test the validity of this hypothesis, we investigated wealth-accumulation trends for different socioeconomic groups and overall inequality using the same sources we used for our earlier analyses. Also, in the course of making our analyses, we developed novel applications of statistical techniques which were hitherto used only in disciplines such as econometry and marketing. One such technique allowed us to quantify the effect of the sustained war campaigns, extreme precipitation events and a severe economic crisis in the context of a periphery town in the Ottoman Empire in the 18th Century.

1.2.4. *Log-linear analysis of mobility.* Extending our earlier collaboration with Prof. Ergene [31, 32, 34], in this paper we further explored Ottoman primary sources by further employing sophisticated quantitative techniques. Here, we used loglinear modeling techniques to analyze intergenerational mobility patterns in the Ottoman Empire which has recently become widely accepted research tools in demographic and mobility research [67]. As it was the case in our earlier joint work, we used probate estate inventories (terekes) to examine intergenerational mobility patterns in eighteenth century Ottoman Kastamonu in north-central Anatolia. The study is based an analysis of the honorary titles of fathers and sons, which we take as indicators of economic and communal status.

Our analysis based on our data showed that in the eighteenth century, Kastamonu was composed of two large population blocs: elite title holders, and lower level title holders together with commoners.

These blocs had noticeably dissimilar patterns of father-to-son title transitions. Our findings also indicated that the structural boundaries between lower-level military class and commoners eroded after the seventeenth century. The fact such an erosion took place at the very time of deteriorating wealth levels, and growing wealth disparities in Kastamonu [34] may not be a coincidence. We concluded that the introverted nature of intergenerational transitions observed among elite military and religious title-holders in the later parts of the century could be a sign of increasingly hereditary mobility patterns among Kastamonu's upper echelons, a phenomenon that other researchers have also observed in different parts of the Empire. Although socio-economic stratification remained prevalent throughout our period, we observed that further research is necessary to find out why we observe the emergence of such a strong inclination among Kastamonu's elite title-holders to avoid status and career diversification in the second half of the eighteenth century.

2. Current Research and Research Agenda

2.1. **Mathematics.**

2.1.1. *Model categories, Goodwillie calculus and derived algebraic geometry.* Since the category of algebras is not an abelian category, any cohomology theory (Hopf-equivariant or not) defined on the category of algebras should be obtained by using abstract homotopy theory and closed model categories [118, 29, 60]. There are notable examples of such an approach in the literature [20, 63, 41]. There are cohomology theories for various types of algebras where suitable Hopf equivariance exist, and Hopf-equivariant variants of these (co)homology theories should be developed à la [125, 13]. These include André-Quillen cohomology [3, 120], Shukla cohomology [128], topological Hochschild cohomology and stable K-theory [30, 9, 36, 116, 27, 28] and topological cyclic cohomology [10].

One possible approach for Hopf equivariant generalized cohomology theories for equivariant (co)algebras, is to pursue a Hopf equivariant Goodwillie calculus [43, 44, 45] as a way of approaching a Hopf-equivariant K-theory. Gabriel's Reconstruction Theorem for schemes [38] and its subsequent generalizations by Rosenberg [123, 124] together with Reconstruction Theorems coming from Tannaka-Krein framework [104, 64], and finally Bondal's Derived Reconstruction Theorem [11] strongly indicate that a derived algebraic geometrical approach is the most appropriate. I believe there is a whole body of work involving Hopf algebras, various categories of spectra, and topological Hochschild and cyclic cohomology waiting to be investigated through the prism of Hopf-equivariance.

2.1.2. *Non-commutative rational homotopy theory.* One of the computational successes of topology is the rational homotopy theory [119, 35]. One can attribute this success to the equivalence between homotopy category of simply connected rational spaces and the category of differential graded commutative rational algebras [130] and the category of differential graded rational Lie algebras [119]. Since its inception, non-commutative geometry has been pursuing the category of arbitrary algebras as the replacement of the category of spaces. In this light, non-commutative homotopy theory should replace the homotopy category of spaces with the homotopy category of arbitrary differential graded algebras. Other possible direction is to use the non-commutative analogue of Lie algebras, namely Leibniz algebras [94]. The only work that I am aware of in this direction is by Livernet [91]. Investigation of different model category structures [118, 60] on the category of differential graded Leibniz algebras will give considerable insight and will probably lead to some success in defining the right analogues of fundamental groups for non-commutative spaces.

2.1.3. *Hopf-equivariant K-theory and equivariant Chern-Connes character.* Now that there is a well-developed theory of Hopf-equivariant cyclic cohomology, one should also investigate the possibility of defining a Hopf-equivariant K-theory and an equivariant Chern-Connes character. Following my previous work on Loday-Quillen-Tsygan Theorem, one possibility is to use Fresse's Freeness Theorem [37], or Loday's generalized operadic framework [92] in the homotopy category of symmetric spectra in conjunction with Waldhausen's K-theory. Of course one will need an H-space (or a cogroup object in the homotopy category of spectra) acting on the K-theory spectrum in a compatible way, replacing the Hopf algebra acting on the underlying algebra.

2.1.4. *Leibniz and associative properads.* There is a sequence of weakenings: PROPs [103] to operads [110] to properads [135]. One can further remove the monoidal structure from the underlying index category whereby weaken the notion of a properad. On this level, one only has the index category and a symmetric structure which need not be the canonical symmetric structure used in the definition of a operad. At this generality I proved a result that the (weak) Leibniz properad and the (weak) symmetric associative properad are isomorphic [71] over a base commutative ring of arbitrary characteristic.

2.1.5. *Noncommutative Grassmanians and flag varieties.* Recently in [51] and [52] Hajac, Zielinski and myself found a novel way in incorporating combinatorial data and topological data together thus paving a way to describe noncommutative spaces both in terms of continuous and discrete parameters based on [55]. In an ongoing project, we are working on applying this method in constructing quantum analogues of complex projective spaces and Grassmanian varieties, or quantum flag varieties in general.

2.1.6. *Brzezinski crossed product algebras and their cohomology.* Quantum enveloping algebras (QEA) of Drinfeld and Jimbo [26] are constructed via a procedure called *double bozonization* by Majid [106]. This procedure is actually is based on writing an algebraic structure as a product of some of its substructures. This construction is also known as a*distributive law* [7] or a *factorization system* [102]. Then one obtains QEAs by deforming the interactions of the pieces. Moscovici and Rangipour also used this strategy to construct their deformation of the diffeomorphism group of $\mathbb{R}$ [115]. These constructions can all be gathered under the heading of Brzezinski crossed product [14]. In our previous collaborations, Sütlü and used a Hochschild-Serre type spectral sequence to calculate Hochschild and cyclic homologies of smashed products between Hopf algebras and their module (co)algebras to perform our calculations [84, 83, 82, 69]. We expect that our framework would extend to Brzezinski products with little or no modification.

2.1.7. *Graph algebras and their quotients.* Graphs provide an ample source for problems in representation theory and homology of algebras [39, 40, 5, 4, 89, 22]. We expect a nontrivial combination of the Hochschild-Serre spectral sequence [58, 59] and Jacobi-Zariski long exact sequence [78] to yield new and interesting results in this direction.

2.2. **Applied statistics.**

2.2.1. *Statistical methods in data analysis in the context of Ottoman history.* My collaboration with Prof. Ergene showed that data mining and statistical analyses of data, which have been used very effectively in data-rich fields such as biology, are still *terra incognita* in history. Our analyses of the effects of precipitation patterns and war events on the local economy of an Ottoman province in the 18th Century proved that such analyses can provide pertinent information on the social, economic and political characteristics which can not be obtained from the available sources by other methods. To fill this rather crucial gap together with Prof. Ergene we were able to develop sophisticated statistical and data mining techniques on data coming from Ottoman court documents. We have a long-term research plan on developing large data-sets for other provinces in the Ottoman empire similar to the data-sets we develop for our earlier studies.

2.3. **Applied mathematics and data analysis.**

2.3.1. *Tropic methods in optimization and data analysis.* Tropicalization is a relatively new area [62, 111, 8], but nevertheless, it already has been effectively used in solving optimization problems such as the shortest path problem [132]. There are well-known algorithms in finding cheapest paths in a weighted graph [23, 117]. These new *tropic* algorithms are not yet as efficient as the state-of-the-art. But they make up for their deficiencies by being more suitable for parallel implementations. As a on-going project, together with my Colleague Prof. Ceyhan from University of Luxembourg, we have been looking into developing new algorithms and implementations of such shortest-path problems and their applications in various fields.

2.3.2. *Combinatorial homotopy theory and neural networks.* An recent intriguing interdisciplinary collaboration of neurobiologists and mathematicians [61, 107] suggests that the interaction between the stimulations and neuronal spiking activity indicates that our brain encodes the topology of the stimuli using combinatorial means. Specifically, Manin's work [107] claims that the homotopy type of the space of stimuli and its covering can be reconstructed from the corresponding neural code. There are also hints that the homotopy theory of neural codes and the information bottleneck theory of artificial neural networks are related [129]. Taken as a whole, these papers indicate that neural networks (both natural and artificial varieties) seem to retain only the most relevant features of inputs when they generate their outputs. In an ongoing project with Prof. Ceyhan, we have been investigating connections between combinatorial homotopy theory and artificial neural networks.

2.3.3. *Shannon entropy, and the Turing test.* Shannon's concept of the entropy as the information content of a communication channel has brought information theory a solid branch of applied mathematics and statistics [126]. In another ongoing project, Prof. Ceyhan and I are investigating different formal ways one can interpret Turing's test [134] as a version of the Kullback-Leibler divergence [87] in which we compare artificial and human agents as distributions with specific information contents.

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