MAT 281E – Linear Algebra and Applications

Fall 2011

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Class Meets : 13.30 – 16.30, Friday
EEB 4104

Office Hours : 10.00 – 12.00, Friday

Textbook : G. Strang, 'Introduction to Linear Algebra', 4^{th} Edition, Wellesley Cambridge.

Grading : Homeworks (10%), Midterm Exam (30%), 2 Quizzes (10% each), Final (40%).

Webpage : `http://ninova.itu.edu.tr/Ders/1039/Sinif/3380`

Tentative Course Outline

- Solving Linear Equations via Elimination

  Linear system of equations, elimination, LU Decomposition, Inverses

- Vector Spaces

  The four fundamental subspaces, solving $Ax = b$, rank, dimension.

- Orthogonality

  Orthogonality, projection, least squares, Gram-Schmidt orthogonalization.

- Determinants

  Determinant, cofactor matrices, Cramer rule.

- Eigenvalues and Eigenvectors

  Eigenvalues, eigenvectors, diagonalization, application to difference equations, symmetric matrices, positive definite matrices, iterative splitting methods for solving linear systems, singular value decomposition.

MAT 281E – Homework 1

Due 07.10.2011

1. Let
$$A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}.$$

   (a) Find two matrices C_1, C_2 so that,
   $$C_1\, A\, C_2 = a + b + c + d + e + f + g + h + i.$$

   (b) Find two matrices P_1, P_2 so that,
   $$P_1\, A\, P_2 = \begin{bmatrix} g & h & i \\ 2\,a & 2\,b & 2\,c \\ d & e & f \end{bmatrix}.$$

   (c) Find two matrices $\tilde{P}_1$, $\tilde{P}_2$ so that,
   $$\tilde{P}_1\, A\, \tilde{P}_2 = \begin{bmatrix} b & c & 2\,a \\ h & i & 2\,g \\ e & f & 2\,d \end{bmatrix}.$$

2. Suppose we know that
$$A \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \qquad A \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \qquad A \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \qquad A \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 3 \\ 5 \\ 2 \end{bmatrix}.$$

   Find x_1, x_2, x_3.

3. Find the solution of
$$\begin{bmatrix} 2 & 1 & 2 & 3 \\ 4 & 2 & 1 & 0 \\ 6 & 5 & 0 & 1 \\ 0 & 2 & 3 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 1 \\ 8 \\ 15 \\ 2 \end{bmatrix}.$$

   Apply elimination on the augmented matrix to find the solution (show your steps). Express the elimination matrices you used for each step. Also, write down the pivots you used.

4. Suppose we know that,
$$\begin{bmatrix} 3 & 0 & 1 \\ 1 & 3 & 2 \\ 0 & 1 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 2 \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}.$$

   Find x_1, x_2, x_3 so that at least one of them is non-zero.

MAT 281E - HW1 Solutions

1. (a) Notice that we need to find the sum of the entries of A.

To get the sum of the rows: $A\begin{bmatrix}1\\1\\1\end{bmatrix} = \begin{bmatrix}\text{sum (row 1)}\\ \text{sum (row 2)}\\ \text{sum (row 3)}\end{bmatrix} = \begin{bmatrix}a+b+c\\ d+e+f\\ g+h+i\end{bmatrix}$

Multiply this with $[1\ 1\ 1]$

we obtain: $[1\ 1\ 1]\,A\begin{bmatrix}1\\1\\1\end{bmatrix} = [\text{sum (row 1)} + \text{sum (row 2)} + \text{sum (row 3)}]$

$= [a+b+c+d+e+f+g+h+i]$.

(b) To get the matrix on the right hand side, we move

$$\left.\begin{array}{l}\cdot \text{ row 1 of } A \text{ to } 2^{nd} \text{ row position and multiply by } 2\\ \cdot \text{ row 2 of } A \text{ to } 3^{rd} \text{ position}\\ \cdot \text{ row 3 of } A \text{ to } 1^{st} \text{ position}\end{array}\right\}\text{ row operations}$$

$\Rightarrow$ if $P_1 = \begin{bmatrix}0&0&1\\2&0&0\\0&1&0\end{bmatrix} \Rightarrow P_1A$ does it. Set $P_2 = I$.

(c) Both row & column operations ...

$$\tilde{P}_1 = \underbrace{\begin{bmatrix}1&0&0\\0&0&1\\0&1&0\end{bmatrix}}_{\substack{\text{exchange row 2}\\ \text{with row 3}}} \quad ; \quad \tilde{P}_2 = \underbrace{\begin{bmatrix}0&0&1\\1&0&0\\0&1&0\end{bmatrix}}_{\substack{\text{move}\\ \text{col 1} \to \text{col 3}\\ \text{col 2} \to \text{col 1}\\ \text{col 3} \to \text{col 2}}}\underbrace{\begin{bmatrix}1&0&0\\0&1&0\\0&0&2\end{bmatrix}}_{\to \text{ multiply col 2 by 2.}}$$

2. Notice $A\begin{bmatrix}2\\1\\1\end{bmatrix} - A\begin{bmatrix}1\\1\\0\end{bmatrix} = A\begin{bmatrix}1\\0\\1\end{bmatrix} = \begin{bmatrix}0\\0\\1\end{bmatrix}$

$$\Rightarrow A\underbrace{\begin{bmatrix}1 & 1 & 1\\1 & -1 & 0\\0 & 0 & 1\end{bmatrix}}_{B} = \underbrace{\begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\end{bmatrix}}_{I} \Rightarrow B = A^{-1}!$$

$$x = A^{-1}b = \begin{bmatrix}1 & 1 & 1\\1 & -1 & 0\\0 & 0 & 1\end{bmatrix}\begin{bmatrix}3\\5\\2\end{bmatrix} = \begin{bmatrix}10\\-2\\2\end{bmatrix}$$

3.
$$\begin{bmatrix}2 & 1 & 2 & 3 & 1\\4 & 2 & 1 & 0 & 8\\6 & 5 & 0 & 1 & 15\\0 & 2 & 3 & 2 & 2\end{bmatrix} \xrightarrow{r_2-2r_1} \begin{bmatrix}2 & 1 & 2 & 3 & 1\\0 & 0 & -3 & -6 & 6\\6 & 5 & 0 & 1 & 15\\0 & 2 & 3 & 2 & 2\end{bmatrix} \xrightarrow{r_3-3r_1} \begin{bmatrix}2 & 1 & 2 & 3 & 1\\0 & 0 & -3 & -6 & 6\\0 & 2 & -6 & -8 & 12\\0 & 2 & 3 & 2 & 2\end{bmatrix}$$

$$\xrightarrow{r_2\leftrightarrow r_3} \begin{bmatrix}2 & 1 & 2 & 3 & 1\\0 & 2 & -6 & -8 & 12\\0 & 0 & -3 & -6 & 6\\0 & 2 & 3 & 2 & 2\end{bmatrix} \xrightarrow{r_4-r_2} \begin{bmatrix}2 & 1 & 2 & 3 & 1\\0 & 2 & -6 & -8 & 12\\0 & 0 & -3 & -6 & 6\\0 & 0 & 9 & 10 & -10\end{bmatrix} \xrightarrow{r_4+3r_3} \begin{bmatrix}2 & 1 & 2 & 3 & 1\\0 & 2 & -6 & -8 & 12\\0 & 0 & -3 & -6 & 6\\0 & 0 & 0 & -8 & 8\end{bmatrix}$$

$$\left.\begin{aligned}
&\Rightarrow x_4 = -1\\
&-3x_3 + 6 = 6 \Rightarrow x_3 = 0\\
&2x_2 - 6\cdot 0 - 8\cdot(-1) = 12 \Rightarrow x_2 = 2\\
&2x_1 + 1\cdot 2 + 2\cdot 0 + 3\cdot(-1) = 1 \Rightarrow x_1 = 1
\end{aligned}\right\} \text{Back-Substitution.}$$

Elimination Matrices:

$r_2 - 2r_1$: $\begin{bmatrix} 1 & 0 & 0 & 0 \\ -2 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ $\qquad$ $r_3 - 3r_1$: $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -3 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

$r_2 \longleftrightarrow r_3$: $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ $\qquad$ $r_4 - r_2$: $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix}$

$r_4 + 3r_3$: $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 3 & 1 \end{bmatrix}$

4. This is an equality of the form $Ax = 2x$. Rewrite it as $(A - 2I)x = 0$. Apply elimination on $A - 2I$:

$$\underbrace{\begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 2 \\ 0 & 1 & 1 \end{bmatrix}}_{A-2I} \longrightarrow \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$\Rightarrow$ We are looking for a non-zero solution of:

$$\left.\begin{array}{l} x_1 + x_3 = 0 \\ x_2 + x_3 = 0 \end{array}\right\} \Rightarrow \text{Set } \boxed{x_3 = 1 \Rightarrow x_1 = -1,\ x_2 = -1.}$$

MAT 281E – Homework 2

Due 28.10.2011

1. Let
$$A = \begin{bmatrix} 2 & 1 & 3 & -1 \\ 4 & 3 & 7 & 0 \\ 0 & 2 & -1 & 3 \\ 2 & 1 & 3 & 1 \end{bmatrix}.$$
Find the LU decomposition of A.

2. Which of the following subsets of $\mathbb{R}^3$ also form subspaces of $\mathbb{R}^3$? Please explain your answers.
   (a) All vectors $\begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix}$ with $x_1 = 0$.
   (b) All vectors $\begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix}$ with $x_2 = 1$.
   (c) The vector $\begin{bmatrix} 1 & 1 & 1 \end{bmatrix}$ alone.
   (d) The vector $\begin{bmatrix} 0 & 0 & 0 \end{bmatrix}$ alone.
   (e) All vectors $\begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix}$ with $x_2^2 - x_3 = 0$.
   (f) All vectors $\begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix}$ with $x_1 + 2\,x_3 = 1$.
   (g) All vectors $\begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix}$ with $2\,x_1 + x_3 = 0$.
   (h) All vectors $\begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix}$ with $x_1 = x_2 = 2\,x_3$.

3. (a) Find a 3×4 matrix A whose column space is the span of
$$v_1 = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} \quad \text{and} \quad v_2 = \begin{bmatrix} 7 \\ -10 \\ 1 \end{bmatrix}.$$
   (b) Find a 3×2 matrix A whose null space is the span of
$$v = \begin{bmatrix} 1 \\ -1 \end{bmatrix}.$$

4. (a) Suppose we are given a matrix A and a column vector b such that $A\,x = b$ has no solution. Let us define the augmented matrix $B = \begin{bmatrix} A & b \end{bmatrix}$ (B has one more column than A). Let $C(A)$ and $C(B)$ denote the column spaces of A and B.
   Which is true in general – '$C(A) \subset C(B)$' or '$C(B) \subset C(A)$'? (If both are true in general, write so.) Please explain your answer.
   (b) Suppose we are given a matrix A and we define the augmented matrix $D = \begin{bmatrix} A & A \end{bmatrix}$ (the matrix A is augmented to A). Let $C(A)$ and $C(D)$ denote the column spaces of A and D.
   Which is true in general – '$C(A) \subset C(D)$' or '$C(D) \subset C(A)$'? (If both are true in general, write so.) Please explain your answer.

5. Let
$$A = \begin{bmatrix} 1 & 1 & 1 & 0 \\ 2 & 2 & 3 & 1 \\ 1 & 1 & 2 & 1 \\ 3 & 3 & 8 & 3 \end{bmatrix}.$$
Describe $N(A)$, the nullspace of A (i.e. find the special solutions to $A\,x = 0$).

MAT281E - HW2 Soln.

(1.)

$$\begin{bmatrix} 2 & 1 & 3 & -1 \\ 4 & 3 & 7 & 0 \\ 0 & 2 & -1 & 3 \\ 2 & 1 & 3 & 1 \end{bmatrix} \to \begin{bmatrix} 2 & 1 & 3 & -1 \\ 0 & 1 & 1 & 2 \\ 0 & 2 & -1 & 3 \\ 2 & 1 & 3 & 1 \end{bmatrix} \to \begin{bmatrix} 2 & 1 & 3 & -1 \\ 0 & 1 & 1 & 2 \\ 0 & 2 & -1 & 3 \\ 0 & 0 & 0 & 2 \end{bmatrix} \to \underbrace{\begin{bmatrix} 2 & 1 & 3 & -1 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & -3 & -1 \\ 0 & 0 & 0 & 2 \end{bmatrix}}_{U}$$

$$E_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -2 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad E_2 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & 1 \end{bmatrix} \quad E_3 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -2 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\Rightarrow E_3 E_2 E_1 A = U \Rightarrow A = \underbrace{E_1^{-1} E_2^{-1} E_3^{-1}}_{L} U$$

$$L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 0 & 2 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix}$$

(2.) a, d, g, h form subspaces. The rest do not.

(3.) (a) $A = \begin{bmatrix} 1 & 7 & 0 & 0 \\ -1 & -10 & 0 & 0 \\ 2 & 1 & 0 & 0 \end{bmatrix}$ (b) $A = \begin{bmatrix} 5 & 5 \\ 2 & 2 \\ 7 & 7 \end{bmatrix}$ ($A = \begin{bmatrix} c & c \end{bmatrix}$ in general.)

(4.) (a) $C(A) \subset C(B)$. To see this, take $d \in C(A)$. This means, $d = Ay$ for some y. But then, $\underbrace{\begin{bmatrix} A & b \end{bmatrix}}_{B} \begin{bmatrix} y \\ 0 \end{bmatrix} = d \Rightarrow d \in C(B)$ also.
To see $C(A) \not\supset C(B)$, notice that $b \in C(B)$ but $b \notin C(A)$ otherwise $Ax = b$ would have a solution.

(b) $C(A) = C(D)$ (i.e. both $C(A) \subset C(D)$ and $C(D) \subset C(A)$ are true).

Since $\begin{bmatrix} A & A \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = Ax_1 + Ax_2 = A(x_1 + x_2)$.

5. $\begin{bmatrix} 1 & 1 & 1 & 0 \\ 2 & 2 & 3 & 1 \\ 1 & 1 & 2 & 1 \\ 3 & 3 & 8 & 3 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 2 & 1 \\ 3 & 3 & 8 & 3 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 3 & 3 & 8 & 3 \end{bmatrix} \longrightarrow$

$\begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 5 & 3 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 5 & 3 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & -2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

Free variable: x_2

Pivot variables: x_1, x_3, x_4

Set $x_2 = 1 \Rightarrow x_4 = 0, \; x_3 = 0, \; x_1 = -1$

$\Rightarrow$ Special soln: $\begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \end{bmatrix} \Rightarrow N(A) = \left\{ \alpha \begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \end{bmatrix} \right\}_{\alpha \in \mathbb{R}}$.

MAT 281E – Homework 3

Due 18.11.2011

1. Is it possible to find a 3×2, non-zero matrix A such that, the set of vectors of the form '$A \begin{bmatrix} x \\ y \end{bmatrix}$', where $x \geq 0$, $y \geq 0$, form a subspace of $\mathbb{R}^3$? If it is possible, provide such a matrix. If you think it is not possible, explain why not.

2. Consider the system of equations

$$\underbrace{\begin{bmatrix} 1 & 0 & 3 & -2 \\ 0 & 0 & 3 & 1 \\ 2 & -1 & 9 & 0 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}}_{x} = \underbrace{\begin{bmatrix} -2 \\ 2 \\ -3 \end{bmatrix}}_{b}.$$

   (a) Describe $N(A)$, the nullspace of A (find the special solutions).

   (b) What is the rank of A?

   (c) What is the dimension of $C(A)$, the column space of A?

   (d) What is the dimension of $N(A)$?

   (e) Describe the solution set of $A\,x = b$ (find a particular solution and use $N(A)$).

3. Find a 3×3 system $A\,x = b$ (i.e. find a 3×3 matrix A and a vector b) whose set of solutions is described by

$$\begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix} + \alpha \begin{bmatrix} 2 \\ 2 \\ -1 \end{bmatrix}$$

   where α can be any real number.

4. Consider the matrix

$$A = \begin{bmatrix} 1 & 0 & 1 & 3 \\ 1 & 1 & 3 & 5 \\ 2 & -1 & 0 & 4 \end{bmatrix}.$$

   Let r be its rank.

   (a) Find r.

   (b) Find a $3 \times r$ matrix B and an $r \times 4$ matrix C such that $B\,C = A$.

5. (a) Find a basis for the plane $x + 2\,y - z = 0$ (i.e. find n linearly independent vectors that span the plane – what is n?).

   (b) Recall that two column vectors v, w are said to be orthogonal if $v^T\,w = 0$. Find a vector u that is orthogonal to any vector in the plane described above.

6. A hyperplane is essentially a 'high-dimensional plane'. For instance, the set of solutions to '$x_1 + 2\,x_2 - x_3 + x_4 = 0$' describes a hyperlane in $\mathbb{R}^4$. Let us call this set P.

   (a) Is P a subspace or not? (Please explain your answer)

   (b) What is the maximum number of linearly independent vectors you can find in P? Provide such a set of vectors.

MAT281E - HW3 Soln.

(1.) Yes, any $A = [c \quad -c]$ work. For ex: $A = \begin{bmatrix} 1 & -1 \\ 1 & -1 \\ 0 & 0 \end{bmatrix}$.

(*) If $d = A\begin{bmatrix} x \\ y \end{bmatrix}$ with $x \geq 0$, $y \geq 0$ $\Rightarrow$ for $\alpha > 0 \Rightarrow \alpha d = A\begin{bmatrix} \alpha x \\ \alpha y \end{bmatrix}$

for $\alpha < 0 \Rightarrow \alpha d = A\begin{bmatrix} |\alpha| y \\ |\alpha| x \end{bmatrix}$

Notice: $|\alpha| y > 0$
$|\alpha| x > 0$

(*) Also, if $d_1 = A\begin{bmatrix} x_1 \\ y_1 \end{bmatrix}$, $d_2 = A\begin{bmatrix} x_2 \\ y_2 \end{bmatrix}$ with $x_1, x_2, y_1, y_2 \geq 0$,

then $d_1 + d_2 = A\begin{bmatrix} x_1 + x_2 \\ y_1 + y_2 \end{bmatrix}$ and $x_1 + x_2 \geq 0$
$y_1 + y_2 \geq 0$

(2.) Let's work with the augmented matrix $[A \vdots b]$.

$$\begin{bmatrix} 1 & 0 & 3 & -2 & -2 \\ 0 & 0 & 3 & 1 & 2 \\ 2 & -1 & 9 & 0 & -3 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & 3 & -2 & -2 \\ 0 & 0 & 3 & 1 & 2 \\ 0 & -1 & 3 & 4 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 3 & -2 & -2 \\ 0 & -1 & 3 & 4 & 1 \\ 0 & 0 & 3 & 1 & 2 \end{bmatrix}$$

(a) Solve $\begin{bmatrix} 1 & 0 & 3 & -2 \\ 0 & -1 & 3 & 4 \\ 0 & 0 & 3 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ 1 \end{bmatrix} = 0 \Rightarrow \left(\begin{array}{l} x_3 = -1/3 \\ x_2 = 3 \\ x_1 = 3 \end{array} \right)$

(x_4 is the free variable)

$\Rightarrow$ special soln: $s_1 = \begin{bmatrix} 3 \\ 3 \\ -1/3 \\ 1 \end{bmatrix}$ $\quad N(A) = \{\alpha \cdot s_1\}_{\alpha \in \mathbb{R}}$.

(b) Rank $= r =$ # of pivot variables $= 3$.

(c) $\dim(C(A)) = 3 =$ # of pivot columns

(d) $\dim(N(A)) = 1 =$ # of free columns.

(e) To find a particular soln, solve $\begin{bmatrix} 1 & 0 & 3 & -2 \\ 0 & -1 & 3 & 4 \\ 0 & 0 & 3 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}$.

$\Rightarrow x_3 = 2/3$

$x_2 = 1$ $\Rightarrow$ particular soln: $y_p = \begin{bmatrix} -4 \\ 1 \\ 2/3 \\ 0 \end{bmatrix}$

$x_1 = -4$

Soln set: $\left\{ \begin{bmatrix} -4 \\ 1 \\ 2/3 \\ 0 \end{bmatrix} + \alpha \cdot \begin{bmatrix} 3 \\ 3 \\ -1/3 \\ 1 \end{bmatrix} \right\}_{\alpha \in \mathbb{R}}$.

3. We need a matrix whose null-space is spanned by $\begin{bmatrix} 2 \\ 2 \\ -1 \end{bmatrix}$.

Take $A = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$. $\Rightarrow A \cdot \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix} = \underbrace{\begin{bmatrix} 4 \\ 3 \\ 0 \end{bmatrix}}_{b}$

$\Rightarrow$ Soln of $Ax=b$ is the set described in the question.

(4.)

$$\begin{bmatrix}1&0&1&3\\1&1&3&5\\2&-1&0&4\end{bmatrix} \longrightarrow \begin{bmatrix}1&0&1&3\\0&1&2&2\\2&-1&0&4\end{bmatrix} \longrightarrow \begin{bmatrix}1&0&1&3\\0&1&2&2\\0&-1&-2&-2\end{bmatrix} \longrightarrow \begin{bmatrix}1&0&1&3\\0&1&2&2\\0&0&0&0\end{bmatrix}$$

pivot columns.

$$E_1 = \begin{bmatrix}1&0&0\\-1&1&0\\0&0&1\end{bmatrix} \qquad E_2 = \begin{bmatrix}1&0&0\\0&1&0\\-2&0&1\end{bmatrix} \qquad E_3 = \begin{bmatrix}1&0&0\\0&1&0\\0&1&1\end{bmatrix}$$

$r = \#$ of pivot columns $= 2$.

$$E_3 E_2 E_1 A = \underbrace{\begin{bmatrix}1&0\\0&1\\0&0\end{bmatrix}}_{D} \underbrace{\begin{bmatrix}1&0&1&3\\0&1&2&2\end{bmatrix}}_{C} \Rightarrow A = \underbrace{\left(E_1^{-1} E_2^{-1} E_3^{-1} \cdot D\right)}_{=B} \cdot C$$

(5.) (a) We need to find a basis for the null-space of $\begin{bmatrix}1 & 2 & -1\end{bmatrix}$.

piv. column   free col.

The special solutions provide such a basis.

$$y_{s_1} = \begin{bmatrix}-2\\1\\0\end{bmatrix}, \quad y_{s_2} = \begin{bmatrix}1\\0\\1\end{bmatrix}.$$

(b) The vector $\begin{bmatrix}1 & 2 & -1\end{bmatrix}$ is orthogonal to these.

(6) (a) P is the null-space of $A = \begin{bmatrix}1 & 2 & -1 & 1\end{bmatrix}$. It is, therefore a subspace.

(b) # of free columns of $A = 3 \Rightarrow 3$ special solutions

$$y_{s_1} = \begin{bmatrix}-2\\1\\0\\0\end{bmatrix}, \quad y_{s_2} = \begin{bmatrix}1\\0\\1\\0\end{bmatrix}, \quad y_{s_3} = \begin{bmatrix}-1\\0\\0\\1\end{bmatrix}.$$

MAT 281E – Homework 4

Due 09.12.2011

1. Let
$$A = \begin{bmatrix} 1 & 2 \\ 1 & 1 \\ 1 & 2 \end{bmatrix}, \quad B = \begin{bmatrix} 2 & 1 \\ 1 & 4 \\ -1 & 2 \end{bmatrix}.$$

   (a) What are the dimensions of $C(A)$ and $C(B)$?

   (b) Is $C(A) \cap C(B) = 0$? If so, explain why. If not, find a non-zero vector in $C(A) \cap C(B)$.

2. (a) Let S_1 be the plane described by the equation $x_1 + x_2 + x_3 = 0$. Find a basis for $S_1^{\perp}$.

   (b) Let S_2 be the plane described by the equation $x_1 + x_2 - x_3 = 0$. Let $S = S_1 \cap S_2$. Find a basis for $S^{\perp}$.

3. Let S be the subspace of $\mathbb{R}^3$ spanned by $\begin{bmatrix} 1 & -1 & 1 \end{bmatrix}$.

   (a) Find the projection of $b = \begin{bmatrix} 2 & 3 & 3 \end{bmatrix}$ onto S.

   (b) Find the projection matrix P that projects any vector onto S.

4. Let S be the plane described by the equation $x_1 + 2x_2 - x_3 = 0$.

   (a) Find the projection matrix P that projects any vector onto S.

   (b) Find the projection matrix P that projects any vector onto $S^{\perp}$.

MAT 201 E - HW4 Solutions

1. (a) dim $C(A) = 2 \rightarrow$ (the columns are independent)

dim $C(B) = 2$

(b) The question should have asked to find a non-zero vector in $C(A) \cap C(B)$.

Take the matrix $D = [A \; B]$. D is $3\times 4 \Rightarrow$ there is a non-zero vector in its nullspace. If $Dx = 0 \Rightarrow \underbrace{A\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}}_{\in C(A)} = \underbrace{-B\begin{bmatrix} x_3 \\ x_4 \end{bmatrix}}_{\in C(B)}$

So let's find a vector in $N(D)$.

$$\begin{bmatrix} 1 & 2 & 2 & 1 \\ 1 & 1 & 1 & 4 \\ 1 & 2 & -1 & 2 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 2 & 2 & 1 \\ 0 & -1 & -1 & 3 \\ 0 & 0 & -3 & 1 \end{bmatrix} \Rightarrow \text{special soln: } \begin{bmatrix} a \\ b \\ 1/3 \\ 1 \end{bmatrix}$$

$$+B\begin{bmatrix} 1 \\ 3 \end{bmatrix} = -3A\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 5 \\ 13 \\ +5 \end{bmatrix} \in C(A) \cap C(B).$$

2. (a) S_1 is the null-space of $A = [1 \; 1 \; 1]$.

$S_1^{\perp} = C(A^T) \Rightarrow A^T = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \Rightarrow \left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\} \Rightarrow$ a basis for $S_1^{\perp}$

(b) S is the nullspace of $B = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & -1 \end{bmatrix}$

$S^{\perp} = C(B^T) \Rightarrow$ Since the col. of B^T are independent, they form a basis for $C(B^T) \Rightarrow \left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} \right\}$

a basis for $S^{\perp}$

(3.) The projection is $\frac{1}{3}\begin{bmatrix}1\\-1\\1\end{bmatrix}\cdot\begin{bmatrix}1 & -1 & 1\end{bmatrix}\begin{bmatrix}2\\3\\3\end{bmatrix} = \begin{bmatrix}2/3\\2/3\\2/3\end{bmatrix}$

This is the projection matrix.

(4.) (a) Find a basis for S: S is the null-space of $A = \begin{bmatrix}1 & 2 & -1\end{bmatrix}$.

Special soln for A: $\underbrace{\begin{bmatrix}-1/2\\1\\0\end{bmatrix}, \begin{bmatrix}1\\0\\1\end{bmatrix}}_{\text{a basis for } S.}$

Let $V = \begin{bmatrix}-1/2 & 1\\1 & 0\\0 & 1\end{bmatrix}$. Then, the projection matrix is

$$P_S = V(V^TV)^{-1}V^T.$$

(Check!) $= \frac{1}{6}\begin{bmatrix}+5 & -2 & 1\\-2 & 2 & 2\\1 & 2 & 5\end{bmatrix}$

(b) $P_{S^\perp} = I - P_S$

But we can also compute it directly: $\begin{bmatrix}1\\2\\-1\end{bmatrix}$ forms a basis for $S^\perp$

$$\Rightarrow P_{S^\perp} = \frac{1}{6}\begin{bmatrix}1\\2\\-1\end{bmatrix}\begin{bmatrix}1 & 2 & -1\end{bmatrix} = \frac{1}{6}\begin{bmatrix}1 & 2 & -1\\2 & 4 & -2\\-1 & -2 & 1\end{bmatrix}$$

Note: It is easier to do part (b) first!

MAT 281E – Homework 5

Due 16.12.2011

1. (a) Find a vector x that minimizes $\|A\,x - b\|$ where

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 3 & 5 & 2 \\ 2 & 4 & 2 \end{bmatrix}, \qquad b = \begin{bmatrix} 4 \\ 1 \\ 2 \end{bmatrix}.$$

   (b) Let x^* be the vector you found in part (a). Find a vector, $\tilde{b}$ (other than b or $A\,x^*$) so that $\|A\,x - \tilde{b}\|$ achieves its minimum when $x = x^*$.

2. Consider the line l that passes through $\begin{bmatrix} 1 & 1 & 1 \end{bmatrix}$ and $\begin{bmatrix} 1 & 2 & 3 \end{bmatrix}$. Find the closest point of l to $\begin{bmatrix} 2 & 1 & -1 \end{bmatrix}$.

3. Consider the plane P described by the equation $x_1 - x_2 + x_3 = 3$. Find the closest point of P to $\begin{bmatrix} 1 & 1 & 1 \end{bmatrix}$.

4. Find the $Q\,R$ decomposition of

$$A = \begin{bmatrix} 2 & 0 & 0 \\ 2 & 2 & 0 \\ 2 & 1 & -3 \end{bmatrix}.$$

MAT281E - HW5 Soln.

1. (a) We need to solve $A^TAx = A^Tb$. $\Rightarrow A^TA = \begin{bmatrix} 14 & 25 & 11 \\ 25 & 45 & 20 \\ 11 & 20 & 9 \end{bmatrix}$

$A^Tb = \begin{bmatrix} 11 \\ 21 \\ 10 \end{bmatrix}$. Normally, I'd do elimination (if I were a computer — I should have chosen the numbers more carefully!)

<u>Here's an alternative solution</u>: Find $N(A^T)$, project b onto $N(A^T)$, subtract that from b to find $\hat{b}$ and solve $Ax = \hat{b}$.

For $N(A^T) \Rightarrow \left(A^T = \right) \begin{bmatrix} 1 & 3 & 2 \\ 2 & 5 & 4 \\ 1 & 2 & 2 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 3 & 2 \\ 0 & -1 & 0 \\ 0 & -1 & 0 \end{bmatrix} \rightarrow \begin{pmatrix} 1 & 3 & 2 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

$N(A^T) = \left\{ \alpha \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix} \right\}_{\alpha \in \mathbb{R}}$.

Projecting onto $N(A^T)$ is easy: $\left[\left(b^T \cdot \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix} \right) \Big/ \underbrace{5}_{\left([-2 \ 0 \ 1] \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix} \right)} \right] \times \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix} = \frac{-6}{5} \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}$

$\hat{b} = P_{C(A)} b = b - \frac{-6}{5} \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 8/5 \\ 1 \\ 16/5 \end{bmatrix}$

Now solve $Ax = \hat{b}$

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 28/5 \\ 3 & 5 & 2 & 1 \\ 2 & 4 & 2 & 16/5 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 1 & 2 & 1 & 8/5 \\ 0 & -1 & -1 & 19/5 \\ 0 & 0 & 0 & 0 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 1 & 0 & -1 & 46/5 \\ 0 & 1 & 1 & -19/5 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$x = \begin{bmatrix} 46/5 \\ -19/5 \\ 0 \end{bmatrix}$$ minimizes $\|Ax - b\|$.

(b) It's not unique. We can add any vector in $N(A)$ to x.

Since $\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} \in N(A) \Rightarrow \tilde{x} = \begin{bmatrix} 46/5 \\ -19/5 \\ 0 \end{bmatrix} + \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$ works.

(2.) ℓ is described as $p_1 + \alpha \cdot (p_2 - p_1)$ where $p_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$, $p_2 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$

We need to minimize $\left\| p_1 + \alpha (p_2 - p_1) - \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix} \right\|$

$\Rightarrow$ minimize $\left\| \underbrace{\begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}}_{c} [\alpha] - \underbrace{\begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix}}_{d} \right\| \Rightarrow$ The minimizing α satisfies

$\underbrace{c^T c}_{=5} \cdot \alpha = \underbrace{c^T d}_{=-4} \Rightarrow \alpha = -\frac{4}{5}$

The point is: $p_1 + \alpha \cdot (p_2 - p_1) = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + \frac{-4}{5} \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1/5 \\ -3/5 \end{bmatrix}$

(3.) P is the solution set of $\begin{bmatrix} 1 & -1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 3$.

$P = \left\{ \underbrace{\begin{bmatrix} 3 \\ 0 \\ 0 \end{bmatrix}}_{\text{particular soln.}} + \underbrace{\alpha_1 \cdot \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} + \alpha_2 \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}}_{\text{Special solutions}} \right\} \Rightarrow P$ consists of vectors of the form

$$\begin{bmatrix} 3 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 & -1 \\ 1 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix}$$

We need to minimize

$$\left\| \begin{bmatrix} 0 & -1 \\ 1 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix} + \begin{bmatrix} 3 \\ 0 \\ 0 \end{bmatrix} - \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\| = \left\| \underbrace{\begin{bmatrix} 0 & -1 \\ 1 & 0 \\ 1 & 1 \end{bmatrix}}_{C} \begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix} - \underbrace{\begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}}_{d} \right\|$$

Solve $\underset{\parallel}{C^TC} \, \alpha = C^T d$

$$\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \end{bmatrix} \Rightarrow \begin{bmatrix} 2 & 1 & 2 \\ 0 & 3/2 & 2 \end{bmatrix} \Rightarrow \begin{matrix} \alpha_2 = 4/3 \\ \alpha_1 = 1/3 \end{matrix}$$

The point on P is: $\begin{bmatrix} 3 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 & -1 \\ 1 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 4/3 \\ 1/3 \end{bmatrix} = \begin{bmatrix} 8/3 \\ 4/3 \\ 5/3 \end{bmatrix}$

4. $u_1 = \begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix}, \quad u_2 = \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix}, \quad u_3 = \begin{bmatrix} 0 \\ 0 \\ -3 \end{bmatrix}.$

$$q_1 = u_1 / \|u_1\| = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} / \sqrt{3}$$

$$\hat{q}_2 = u_2 - (u_2^T q_1) \cdot q_1 = \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix} - \frac{1}{3} \cdot 3 \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$$

$$q_2 = \hat{q}_2 / \|\hat{q}_2\| = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} / \sqrt{2}$$

$$\hat{q}_3 = u_3 - (u_3^T q_1) q_1 - (u_3^T q_2) q_2 = \begin{bmatrix} 0 \\ 0 \\ -3 \end{bmatrix} - \frac{1}{3}(-3) \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} - 0 \cdot q_2 = \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix}$$

$$q_3 = \hat{q}_3 / \|\hat{q}_3\| = \frac{1}{\sqrt{6}} \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix}$$

$$A = \begin{bmatrix} q_1 & q_2 & q_3 \end{bmatrix} \begin{bmatrix} 2\sqrt{3} & \sqrt{3} & -\sqrt{3} \\ 0 & \sqrt{2} & 0 \\ 0 & 0 & \sqrt{6} \end{bmatrix}$$

MAT 281E, Fall 2011, Quiz 1

Student Name : ____________________

Student Num. : ____________________

1. Consider the linear system of equations :

$$\underbrace{\begin{bmatrix} 1 & 1 & -1 & 3 \\ 2 & 2 & 0 & 4 \\ 0 & 2 & 0 & 1 \\ -2 & 2 & 2 & -3 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}}_{\mathbf{x}} = \underbrace{\begin{bmatrix} 1 \\ 4 \\ -1 \\ -3 \end{bmatrix}}_{\mathbf{b}}$$

Solve for $\mathbf{x}$ using elimination and back-substitution. Use the augmented matrix $\begin{bmatrix} A & \mathbf{b} \end{bmatrix}$. Show your steps clearly.

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2. Let

$$A = \begin{bmatrix} 2 & 1 & 0 \\ 4 & 1 & 2 \\ 0 & -3 & 5 \end{bmatrix}$$

Find the inverse of A by Gauss-Jordan elimination on the augmented matrix $\begin{bmatrix} A & I \end{bmatrix}$.

(**Optional Bonus :** Write down the elimination matrix E you used in the first step of elimination. Also, write down E^{-1}.)

MAT 281E – Linear Algebra and Applications

Midterm Examination

25.11.2011

5 Questions, 120 Minutes

(20 pts) 1. Find the LU decomposition of

$$A = \begin{bmatrix} 1 & -2 & 0 \\ 3 & -7 & 3 \\ 0 & 2 & -4 \end{bmatrix}.$$

(30 pts) 2. Consider the system of equations

$$\underbrace{\begin{bmatrix} 1 & 2 & -1 & 1 & 1 \\ 2 & 3 & -1 & 0 & 1 \\ 1 & 2 & -1 & 1 & 1 \\ -1 & 1 & -2 & 1 & 0 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix}}_{x} = \underbrace{\begin{bmatrix} 2 \\ 3 \\ 2 \\ -3 \end{bmatrix}}_{b}.$$

(a) Describe $N(A)$, the nullspace of A.

(b) What is the rank of A?

(c) What is the dimension of $N(A)$?

(d) Describe the solution set of $A\,x = b$.

(15 pts) 3. Consider the set of solutions to '$2\,x_1 - x_2 + x_3 + 3\,x_4 = 0$' in $\mathbb{R}^4$. Let us call this set P.

(a) Is P a subspace or not? (Please explain your answer)

(b) Find a basis for P.

(15 pts) 4. Let $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$ be a collection of linearly independent column vectors in $\mathbb{R}^n$. Also let the vectors $\mathbf{b}$, $\mathbf{d}$ be defined as,

$$\mathbf{b} = \alpha_1\,\mathbf{v}_1 + \alpha_2\,\mathbf{v}_2 + \ldots + \alpha_k\,\mathbf{v}_k,$$
$$\mathbf{d} = \big(-\alpha_1\,\mathbf{v}_1\big) + \big(-\alpha_2\,\mathbf{v}_2\big) + \ldots + \big(-\alpha_k\,\mathbf{v}_k\big),$$

where each of $\alpha_1, \alpha_2, \ldots, \alpha_k$ is a non-zero real number. Suppose we form the matrices V and U as,

$$V = \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \ldots & \mathbf{v}_k \end{bmatrix}, \quad U = \begin{bmatrix} \mathbf{b} & \mathbf{d} & \mathbf{v}_1 & \mathbf{v}_2 & \ldots & \mathbf{v}_k \end{bmatrix}$$

so that V is an $n \times k$ matrix and U is an $n \times (k+2)$ matrix.

(a) What are the dimensions of the nullspace, column space, row space and the left nullspace of V?

(b) What are the dimensions of the nullspace, column space, row space and the left nullspace of U?

(c) Which columns of V are pivot columns?

(d) Which columns of U are pivot columns?

Please briefly explain your answers for full credit.

(20 pts) 5. Find a 3×3 matrix A, whose nullspace is the span of

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} \quad \text{and} \quad \mathbf{v}_2 = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}.$$

MAT 281E, Fall 2011, Quiz 2

Student Name : ____________________

Student Num. : ____________________

1. (a) Find a vector x that minimizes $\|A\,x - b\|$ where

$$A = \begin{bmatrix} 1 & 1 & 2 \\ 0 & -1 & -1 \\ 1 & 0 & 1 \end{bmatrix}, \qquad b = \begin{bmatrix} -2 \\ -1 \\ 3 \end{bmatrix}.$$

   (b) Let x^* be the vector you found in part (a). Find a vector c (other than b or $A\,x^*$) so that $\|A\,x - c\|$ achieves its minimum when $x = x^*$.

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2. Find the $Q\,R$ decomposition of

$$A = \begin{bmatrix} 3 & -6 & 2 \\ 0 & 2 & -3 \\ 4 & -8 & 11 \end{bmatrix}.$$

MAT 281E – Linear Algebra and Applications

Final Examination

12.01.2012

Student Name : ____________________

Student Num. : ____________________

5 Questions, 120 Minutes

Please Show Your Work!

(25 pts) 1. Consider the system of equations

$$\underbrace{\begin{bmatrix} 1 & 2 & 1 & 0 \\ 2 & 4 & 3 & -2 \\ -1 & -2 & 1 & -4 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}}_{\mathbf{x}} = \underbrace{\begin{bmatrix} 1 \\ 1 \\ -3 \end{bmatrix}}_{\mathbf{b}}.$$

(a) Describe the solution set of $A\,\mathbf{x} = \mathbf{b}$.

(b) What is the rank of A? What are the dimensions of the four fundamental subspaces, $N(A)$, $C(A)$, $N(A^T)$, $C(A^T)$?

(20 pts) 2. Consider the system of equations $A\,\mathbf{x} = \mathbf{b}$, where

$$\mathbf{b} = \begin{bmatrix} 2 \\ 1 \\ 0 \\ -1 \end{bmatrix}.$$

Suppose that the solution set consists of all vectors of the form '$\mathbf{y} + \alpha_1\,\mathbf{u}_1 + \alpha_2\,\mathbf{u}_2$', where α_1 and α_2 are arbitrary real numbers and

$$\mathbf{y} = \begin{bmatrix} 1 \\ 3 \\ -1 \end{bmatrix}, \quad \mathbf{u}_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \quad \mathbf{u}_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}.$$

(a) Find a basis for $N(A)$, the nullspace of A.

(b) What are the dimensions of the four fundamental subspaces, $N(A)$, $C(A)$, $N(A^T)$, $C(A^T)$?

(c) Determine A.

(20 pts) 3. Let S be the subspace of $\mathbb{R}^4$ described by the equation '$x_1 - x_2 + x_3 - 2\,x_4 = 0$'.

(a) Find a basis for S.

(b) Find an orthonormal basis for S.

(20 pts) 4. Let S be a 2-dimensional subspace of $\mathbb{R}^3$. Also, let P_S be the projection matrix for S. Suppose that, for

$$\mathbf{x} = \begin{bmatrix} 3 \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{y} = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix},$$

we have $P_S\,\mathbf{x} = \mathbf{y}$ (i.e. $\mathbf{y}$ is the projection of $\mathbf{x}$ onto S).

(a) Find a basis for $S^\perp$, the orthogonal complement of S.
(Hint : What is the dimension of $S^\perp$?)

(b) Find a basis for S.

(c) Find three linearly independent eigenvectors and the associated eigenvalues for P_S. Briefly explain your reasoning for full credit.

(15 pts) 5. Let A be a matrix with eigenvalues 1, 2, and associated eigenvectors

$$\mathbf{e}_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \quad \mathbf{e}_2 = \begin{bmatrix} 3 \\ 2 \end{bmatrix}.$$

Also, let I denote the 2×2 identity matrix. Compute $(A - I)^{10}$.

(Hint : Think about the eigenvalues and eigenvectors of $A - I$.)