MAT 281E – Linear Algebra and Applications

Fall 2010

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Class Meets : 13.30 – 16.30, Friday
EEB 4104

Office Hours : 10.00 – 12.00, Friday

Textbook : G. Strang, 'Introduction to Linear Algebra', 4^{th} Edition, Wellesley Cambridge.

Grading : Homeworks (10%), 2 Midterms (25% each), Final (40%).

Webpage : http://web.itu.edu.tr/ibayram/Courses/MAT281E/

Tentative Course Outline

- Solving Linear Equations via Elimination

  Linear system of equations, elimination, LU Decomposition, Inverses

- Vector Spaces

  The four fundamental subspaces, solving $Ax = b$, rank, dimension.

- Orthogonality

  Orthogonality, projection, least squares, Gram-Schmidt orthogonalization.

- Determinants

  Determinant, cofactor matrices, Cramer rule.

- Eigenvalues and Eigenvectors

  Eigenvalues, eigenvectors, diagonalization, application to differential/difference equations, symmetric matrices, positive definite matrices, singular value decomposition.

MAT 281E – Homework 1

Due 08.10.2010

1. Consider the linear system of equations,

$$\underbrace{\begin{bmatrix} 1 & 1 & -1 & 0 \\ 0 & 1 & 1 & -1 \\ -1 & 0 & 1 & 1 \\ 1 & -1 & 0 & 1 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}}_{\mathbf{x}} = \underbrace{\begin{bmatrix} \pi \\ \pi \\ \pi \\ \pi \end{bmatrix}}_{\mathbf{b}}$$

   (a) For A, what is the sum of the elements in row 1? row 2? row 3? row 4?

   (b) Find an $\mathbf{x}$ that satisfies the system above.

2. Consider the linear system of equations,

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 4 & 8 \\ 1 & 3 & 9 & 27 \\ 1 & 4 & 16 & 64 \end{bmatrix} \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}}_{\mathbf{x}} = \begin{bmatrix} 2 \\ 3 \\ 4 \\ 5 \end{bmatrix}$$

   Find an $\mathbf{x}$ that satisfies the system above.

3. Let us say that an $n \times n$ matrix with integer entries has property-M if all its rows, columns and diagonals add to the same number and all of its entries are distinct. For example, for $n = 3$, a matrix that has property-M is,

$$\begin{bmatrix} 8 & 3 & 4 \\ 1 & 5 & 9 \\ 6 & 7 & 2 \end{bmatrix}. \tag{1}$$

   Notice that all of its rows, columns and diagonals add to 15.

   Suppose now that A is a 4×4 matrix with entries $\{2, 3, \ldots, 17\}$ and it has property-M. What is the sum of one of its rows?

4. Let A be a 5×5 matrix. Write down the matrix B (multiplying A on the left) that subtracts $3 \times \text{row}_2$ from row_4 and leaves the rest of the rows unchanged. What is B^{-1}?

5. Consider the equation $A\,B = C$ where

$$A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}, \qquad C = \begin{bmatrix} 3a & 4b & 5c \\ 3d & 4e & 5f \\ 3g & 4h & 5i \end{bmatrix}.$$

   (a) Find B.

   (b) Compute $B\,A$.

   (c) Write, *in words*, the action of B when it multiplies A on the right (i.e. how $A\,B$ relates to A); on the left (i.e. how $B\,A$ relates to A).

MAT 281E – HW1 Solutions

1. (a) $\sum \text{row } 1 = 1 + 1 + (-1) + 0 = 1.$

Similarly $\sum \text{row } 2 = \sum \text{row } 3 = \sum \text{row } 4 = 1.$

(b) Recall from one of the examples we did in class that

$$A \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} \sum \text{row } 1 \\ \sum \text{row } 2 \\ \sum \text{row } 3 \\ \sum \text{row } 4 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \Rightarrow \text{Multiply both sides by } \pi \Rightarrow A \underbrace{\begin{bmatrix} \pi \\ \pi \\ \pi \\ \pi \end{bmatrix}}_{X} = \begin{bmatrix} \pi \\ \pi \\ \pi \\ \pi \end{bmatrix}$$

2. Think about the matrix-vector multiplication as a linear combination of the columns of the matrix ("column picture").

$$\Rightarrow x_1 \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} + x_2 \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix} + x_3 \begin{bmatrix} 1 \\ 4 \\ 9 \\ 16 \end{bmatrix} + x_4 \begin{bmatrix} 1 \\ 8 \\ 27 \\ 64 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \\ 4 \\ 5 \end{bmatrix} \Rightarrow \begin{pmatrix} x_1 = 1 \\ x_2 = 1 \\ x_3 = 0 \\ x_4 = 0 \end{pmatrix}$$

solves the system

3. Because A has property-M (it is a "magic matrix"), we have

$$A \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} \sum \text{row } 1 \\ \sum \text{row } 2 \\ \sum \text{row } 3 \\ \sum \text{row } 4 \end{bmatrix} = \begin{bmatrix} c \\ c \\ c \\ c \end{bmatrix}$$ for some 'c'.

Now $$\begin{bmatrix} 1 & 1 & 1 & 1 \end{bmatrix} A \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \sum_{i=1}^{4} \sum_{j=1}^{4} a_{ij}$$ But it is also equal to $\begin{bmatrix} 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} c \\ c \\ c \\ c \end{bmatrix} = 4c.$

Even though we don't know a_{ij} for a particular i, j, we know that $\sum_i \sum_j a_{ij} = 2+3+4+ \dots + 17 = \frac{17 \cdot 18}{2} - 1 = 152 = 4c$

$\Rightarrow \boxed{c = 38}$

4. $B = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & -3 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$ $\quad B^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 3 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$

$\Downarrow$ (add back 3·row 2 to row 4)

5. (a) $B = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 5 \end{bmatrix}$

(b) $BA = \begin{bmatrix} 3a & 3b & 3c \\ 4d & 4e & 4f \\ 5g & 5h & 5i \end{bmatrix}$

(c) B on the right (AB): Multiplies the k^{th} column of A by the k^{th} diagonal entry of B

B on the left (BA): Multiplies the k^{th} row of A by the k^{th} diagonal entry of B.

MAT 281E – Homework 2

Due 22.10.2010

1. Let
$$A = \begin{bmatrix} -1 & -2 & 0 & 1 \\ 0 & 1 & 1 & -3 \\ -2 & 3 & 1 & 2 \\ 0 & -1 & -1 & 6 \end{bmatrix}.$$
Find the LU decomposition of A.

2. Let
$$A = \begin{bmatrix} 0 & -2 & 0 \\ -2 & 1 & -1 \\ 2 & -3 & 2 \end{bmatrix} \quad B = \begin{bmatrix} 1 & -1 & 1 \\ -1 & -3 & -4 \\ -3 & 7 & -1 \end{bmatrix} \quad C = \begin{bmatrix} 0 & 1 & 1 \\ 1 & -1 & 0 \\ -2 & 0 & 1 \end{bmatrix}$$
   (a) Using Gauss-Jordan elimination, find the inverse of A.
   (b) Using Gauss-Jordan elimination, find the matrix D such that $B\,D = C$.
   (Hint : Do not use the inverse of B. Use an augmented matrix of the form $\begin{bmatrix} B & V \end{bmatrix}$ where V is a 3×3 matrix. What should V be?)

3. Let
$$A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}.$$
   (a) Find two permutation matrices P_1, P_2 such that,
$$P_1\,A\,P_2 = \begin{bmatrix} g & h & i \\ a & b & c \\ d & e & f \end{bmatrix}.$$
   (b) Find two permutation matrices $\tilde{P}_1$, $\tilde{P}_2$ such that,
$$\tilde{P}_1\,A\,\tilde{P}_2 = \begin{bmatrix} b & c & a \\ e & f & d \\ h & i & g \end{bmatrix}.$$

4. (a) Suppose we are given a matrix A and $B = \begin{bmatrix} A & b \end{bmatrix}$ where b is a column vector (B has one more column than A). Let $C(A)$ and $C(B)$ denote the column spaces of A and B.
   Which is true in general – '$C(A) \subset C(B)$' or '$C(B) \subset C(A)$'? (If both are true in general, write so.) Please explain your answer.
   (b) Let A, B be given matrices and $D = \begin{bmatrix} A & A\,B \end{bmatrix}$ (the matrix AB is augmented to A). Let $C(A)$ and $C(D)$ denote the column spaces of A and D.
   Which is true in general – '$C(A) \subset C(D)$' or '$C(D) \subset C(A)$'? (If both are true in general, write so.) Please explain your answer.

5. Let x, y, z be vectors such that $x + y + z = 0$. Show that x and y span the same space as y and z. (Hint : Let A denote the space spanned by x and y and B denote the space spanned by y and z. Pick an element from A, show that it is in B. This implies that $A \subset B$(Why?). Then pick an element from B, show that it is in A. This implies that $B \subset A$. If $A \subset B$ and $B \subset A$ then it must be that $A = B$.)

MAT 281E — HW2 Solutions

1.

$$\begin{bmatrix} -1 & -2 & 0 & 1 \\ 0 & 1 & 1 & -3 \\ -2 & 3 & 1 & 2 \\ 0 & -1 & -1 & 6 \end{bmatrix} \xrightarrow[(\text{row}3 - 2\cdot\text{row}1)]{} \begin{bmatrix} -1 & -2 & 0 & 1 \\ 0 & 1 & 1 & 3 \\ 0 & 7 & 1 & 0 \\ 0 & -1 & -1 & 6 \end{bmatrix} \xrightarrow[\substack{(r_3 - 7r_2) \\ (r_4 + r_2)}]{} \underbrace{\begin{bmatrix} -1 & -2 & 0 & 1 \\ 0 & 1 & 1 & 3 \\ 0 & 0 & -6 & -21 \\ 0 & 0 & 0 & 3 \end{bmatrix}}_{U}$$

$$\Downarrow \qquad\qquad\qquad\qquad \Downarrow$$

$$L_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -2 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad L_2 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -7 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}$$

$$L_2 L_1 A = U \Rightarrow A = \underbrace{(L_1^{-1} L_2^{-1})}_{=L} \cdot U$$

$$L = L_1^{-1} L_2^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 2 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 7 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 2 & 7 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix}$$

(This multiplication can actually be performed without multiplying anything — see the book for a discussion.)

2. (a) Augmented Matrix:

$$[A \;\; I] = \left[\begin{array}{ccc|ccc} 0 & -2 & 0 & 1 & 0 & 0 \\ -2 & 1 & -1 & 0 & 1 & 0 \\ 2 & -3 & 2 & 0 & 0 & 1 \end{array}\right] \xrightarrow[\text{row exchange}]{} \left[\begin{array}{ccc|ccc} -2 & 1 & -1 & 0 & 1 & 0 \\ 0 & -2 & 0 & 1 & 0 & 0 \\ 2 & -3 & 2 & 0 & 0 & 1 \end{array}\right]$$

$$\xrightarrow[r_3 + r_1]{} \left[\begin{array}{ccc|ccc} -2 & 1 & -1 & 0 & 1 & 0 \\ 0 & -2 & 0 & 1 & 0 & 0 \\ 0 & -2 & 1 & 0 & 1 & 1 \end{array}\right] \xrightarrow[r_3 - r_2]{} \left[\begin{array}{ccc|ccc} -2 & 1 & -1 & 0 & 1 & 0 \\ 0 & -2 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 & 1 & 1 \end{array}\right]$$

$$\xrightarrow[r_1 + r_3]{} \left[\begin{array}{ccc|ccc} -2 & 1 & 0 & -1 & 2 & 1 \\ 0 & -2 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 & 1 & 1 \end{array}\right] \xrightarrow[r_1 + \frac{r_2}{2}]{} \left[\begin{array}{ccc|ccc} -2 & 0 & 0 & -1/2 & 2 & 1 \\ 0 & -2 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 & 1 & 1 \end{array}\right] \xrightarrow[\substack{r_1/(-2) \\ r_2/(-2)}]{} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1/4 & -1 & -1/2 \\ 0 & 1 & 0 & -1/2 & 0 & 0 \\ 0 & 0 & 1 & -1 & 1 & 1 \end{array}\right]$$

(b) Since $BD = C$, we have $B^{-1}BD = B^{-1}C \Rightarrow D = B^{-1}C$.

Elimination on the augmented matrix $[B \ V]$ is equivalent to multiplying it on the left by B^{-1}, which gives $B^{-1}[B \ V] = [I \ \ B^{-1}V]$ so if we set $V = C$, we can obtain D by Gauss-Jordan elimination.

$$[B \ C] = \left[\begin{array}{ccc|ccc} 1 & -1 & 1 & 0 & 1 & 1 \\ -1 & -3 & -4 & 1 & -1 & 0 \\ -3 & 7 & -1 & -2 & 0 & 1 \end{array}\right] \xrightarrow[r_3 + 3r_1]{r_2 + r_1} \left[\begin{array}{ccc|ccc} 1 & -1 & 1 & 0 & 1 & 1 \\ 0 & -4 & -3 & 1 & 0 & 1 \\ 0 & 4 & 2 & -2 & 3 & 4 \end{array}\right]$$

$$\xrightarrow[r_3 + r_2]{} \left[\begin{array}{ccc|ccc} 1 & -1 & 1 & 0 & 1 & 1 \\ 0 & -4 & -3 & 1 & 0 & 1 \\ 0 & 0 & -1 & -1 & 3 & 5 \end{array}\right] \xrightarrow[\substack{r_1 + r_3 \\ r_2 - 3r_3}]{} \left[\begin{array}{ccc|ccc} 1 & -1 & 0 & -1 & 4 & 6 \\ 0 & -4 & 0 & 4 & -9 & -14 \\ 0 & 0 & -1 & -1 & 3 & 5 \end{array}\right] \xrightarrow[r_1 - \frac{r_2}{4}]{} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -2 & \frac{25}{4} & \frac{38}{4} \\ 0 & -4 & 0 & 4 & -9 & -14 \\ 0 & 0 & -1 & -1 & 3 & 5 \end{array}\right]$$

$$\xrightarrow[\substack{r_2/(-4) \\ r_1/(-1)}]{} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -2 & 25/4 & 19/2 \\ 0 & 1 & 0 & -1 & 9/4 & 7/2 \\ 0 & 0 & 1 & 1 & -3 & -5 \end{array}\right], \quad D = \underbrace{\begin{bmatrix} -2 & 25/4 & 19/2 \\ -1 & 9/4 & 7/2 \\ 1 & -3 & -5 \end{bmatrix}}_{D}$$

3. PA permutes rows of A; AP permutes columns of A.

(a) $P_1 = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$, $P_2 = I$. (b) $\tilde{P}_1 = I$, $P_2 = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$

4. (a) $C(A) \subset C(B)$. Because $C(A)$ = all vectors of the form $\alpha_1 a_1 + \alpha_2 a_2 + \dots + \alpha_n a_n$ where α_i are real numbers and a_j's are the columns of A. But $C(B)$ = all vectors of the form $\alpha_1 a_1 + \alpha_2 a_2 + \dots + \alpha_n a_n + \alpha b$ (setting $\alpha = 0$, we can recover any vector from $C(A)$.

(b) Both are true. $C(A) = C(D)$. Because the columns of AB are linear combinations of the columns of A.

5. Let $p \in A \Rightarrow$ we can find α, β such that $p = \alpha x + \beta y$. But $x = -y - z$, so $p = \alpha(-y - z) + \beta y = (-\alpha)z + (\beta - \alpha)y$ $\Rightarrow p \in B$. So $B \supset A$.

Now let $q \in B$. We can find γ, η s.t. $q = \gamma y + \eta z$. But $z = -y - x$ $\Rightarrow q = (-\eta)x + (\gamma - \eta)y \Rightarrow q \in A$. Thus $A \supset B$. $A \supset B$ together with $B \supset A$ implies $A = B$.

MAT 281E – Homework 3

Due 01.11.2010

1. Which of the following subsets of $\mathbb{R}^3$ also form subspaces of $\mathbb{R}^3$? Please explain your answer.
   (a) All vectors $\begin{pmatrix} x_1 & x_2 & x_3 \end{pmatrix}$ with $x_2 = 0$.
   (b) All vectors $\begin{pmatrix} x_1 & x_2 & x_3 \end{pmatrix}$ with $x_1 = 1$.
   (c) The vector $\begin{pmatrix} 0 & 0 & 0 \end{pmatrix}$ alone.
   (d) All vectors $\begin{pmatrix} x_1 & x_2 & x_3 \end{pmatrix}$ with $x_2\, x_3 = 0$.
   (e) All vectors $\begin{pmatrix} x_1 & x_2 & x_3 \end{pmatrix}$ with $x_2 + x_3 = 1$.
   (f) All vectors $\begin{pmatrix} x_1 & x_2 & x_3 \end{pmatrix}$ with $x_1 + 2\, x_3 = 0$.

2. Consider the system of equations

$$\underbrace{\begin{bmatrix} 1 & 0 & 3 & -2 \\ 0 & 0 & 3 & 1 \\ 1 & 3 & 1 & 3 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}}_{x} = \underbrace{\begin{bmatrix} 6 \\ 2 \\ 5 \end{bmatrix}}_{b}.$$

   (a) Describe $N(A)$, the nullspace of A (find the special solutions).
   (b) What is the rank of A?
   (c) What is the dimension of $N(A)$?
   (d) Describe the solution set of $A\,x = b$ (find a particular solution and use $N(A)$).

3. Find a 2×3 system $A\,x = b$ (i.e. find a 2×3 matrix A and a vector b) whose set of solutions is described by

$$\begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} + \alpha \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$$

   where α can be any real number.

4. Let A be an $m \times n$ matrix with full row rank. If the nullspace of A consists of

$$\alpha \begin{bmatrix} 1 \\ 1 \\ 0 \\ 2 \end{bmatrix},$$

   where α is an arbitrary scalar, what is m and n? Provide such a matrix A.

5. Suppose A is a $5 \times k$ matrix with $k \neq 5$ and it has full column rank. In this case, $C(A)$ is a subset of $\mathbb{R}^5$. Is it possible, for some choice of A and k, that actually $C(A) = \mathbb{R}^5$? If you think it is possible, provide an example. If not, explain why not.

MAT 251E - HW3 Solutions

1. (a) It is a subspace. (i) Sum of two vectors remain within the set.

(i.e. $(x_1 \ 0 \ x_3) + (y_1 \ 0 \ y_3) = (z_1 \ 0 \ z_3)$)

(ii) Multiplying by a scalar $\alpha(x_1 \ 0 \ x_3) = (\alpha x_1 \ 0 \ \alpha x_3)$ does not give a vector outside the described set.

(b) Not a subspace. Take $x = (1 \ 1 \ 1)$, $2x = (2 \ 2 \ 2)$, not in the described set.

(c) Forms a subspace (discussed in class).

(d) Not a subspace. Take $\begin{cases} y = (1 \ 0 \ 1) \\ z = (1 \ 1 \ 0) \end{cases} \Rightarrow y + z = (2 \ 1 \ 1)$

$(y+z)_2 \cdot (y+z)_3 = 1 \neq 0.$

(e) Not a subspace. Same counterexample as (d) works for this case too.

(f) It is a subspace (It is the null-space of the 1×3 matrix $[1 \ 0 \ 2]$).

2. (a) Let us do elimination on the augmented matrix (we'll need this in (d) anyway).

$$\left[\begin{array}{cccc|c} 1 & 0 & 3 & -2 & 6 \\ 0 & 0 & 3 & 1 & 2 \\ 1 & 3 & 1 & 3 & 5 \end{array}\right] \xrightarrow[r_3 - r_1]{} \left[\begin{array}{cccc|c} 1 & 0 & 3 & -2 & 6 \\ 0 & 0 & 3 & 1 & 2 \\ 0 & 3 & -2 & 5 & -1 \end{array}\right] \xrightarrow[\text{exchange rows}]{} \left[\begin{array}{cccc|c} 1 & 0 & 3 & -2 & 6 \\ 0 & 3 & -2 & 5 & -1 \\ 0 & 0 & 3 & 1 & 2 \end{array}\right]$$

$$\xrightarrow{\substack{r_1 - r_3 \\ r_2 + r_3 \cdot 2/3}} \left[\begin{array}{cccc|c} 1 & 0 & 0 & -3 & 4 \\ 0 & 3 & 0 & 17/3 & -1/3 \\ 0 & 0 & 3 & 1 & 2 \end{array}\right] \xrightarrow{\substack{r_2/3 \\ r_3/3}} \left[\begin{array}{cccc|c} 1 & 0 & 0 & -3 & 4 \\ 0 & 1 & 0 & 17/9 & -1/9 \\ 0 & 0 & 1 & 1/3 & 2/3 \end{array}\right]$$

$\underbrace{\text{columns 1, 2, 3}}_{\text{pivot columns}}$; column 4: free column

1 free column $\Rightarrow$ 1 special soln: $I\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} -3 \\ 17/9 \\ 1/3 \end{bmatrix} \cdot x_4 = 0 \Rightarrow$ set $x_4 = 1$ to obtain $y_s = \begin{bmatrix} 3 \\ -17/9 \\ -1/3 \\ 1 \end{bmatrix}$

$\Rightarrow N(A) = $ (Set of all vectors of the form $\alpha \cdot y_s$) where α is an arbitrary scalar

$= \{\alpha \cdot y_s : \alpha \in \mathbb{R}\}.$

(b) Rank of $A = \#$ of pivot columns $= 3$.

(c) Dimension of $N(A) = 1$. (1 free variable $\Rightarrow$ 1 special soln.)

(d) Find the particular soln. by setting $x_4 = 0$ in

$$\begin{bmatrix} 1 & 0 & 0 & -3 \\ 0 & 1 & 0 & 17/9 \\ 0 & 0 & 1 & 1/3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 4 \\ -1/9 \\ 2/3 \end{bmatrix} \Rightarrow y_p = \begin{bmatrix} 4 \\ -1/9 \\ 2/3 \\ 0 \end{bmatrix}$$

Solution set $=$ (All vectors of the form $y_p + \alpha y_s$ where α is an arbitrary scalar)

3. Set of solutions $= \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} + \frac{1}{2} \cdot \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} + \gamma \begin{bmatrix} 1/2 \\ 1/2 \\ 1 \end{bmatrix} = \begin{bmatrix} 3/2 \\ 5/2 \\ 0 \end{bmatrix} + \gamma \begin{bmatrix} 1/2 \\ 1/2 \\ 1 \end{bmatrix}$

where γ is an arbitrary scalar. $\Rightarrow$ particular soln $= \begin{bmatrix} 3/2 \\ 5/2 \\ 0 \end{bmatrix}$, special soln: $\begin{bmatrix} 1/2 \\ 1/2 \\ 1 \end{bmatrix}$

$$\Rightarrow \begin{bmatrix} 1 & 0 & -1/2 \\ 0 & 1 & -1/2 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 3/2 \\ 5/2 \\ 0 \end{bmatrix}$$ is such a system.

4. $N(A) \subset \mathbb{R}^4 \Rightarrow \#$ of columns $= \boxed{4 = n.}$

There's only 1 special solution $\Rightarrow \#$ of free variables $= 1$.

But $\#$ of free variables $= \#$ of variables $-$ rank $= 4 -$ rank

$\Rightarrow$ rank $= 3 = \#$ of rows, (because A has full row rank) $\Rightarrow \boxed{m = 3.}$

An example of such an A matrix is: $\begin{bmatrix} 1 & 0 & 0 & -1/2 \\ 0 & 1 & 0 & -1/2 \\ 0 & 0 & 1 & 0 \end{bmatrix}$

(Think of the null-space as $\gamma \begin{bmatrix} 1/2 \\ 1/2 \\ 0 \\ 1 \end{bmatrix}$)

5. It is not possible that $C(A) = \mathbb{R}^5$.

Since rank $\leq$ min (# of rows, # of columns), $k \neq 5$ and rank $= k$,

we get $k < 5$. A basis for $\mathbb{R}^5$ is $\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$, which implies that $\mathbb{R}^5$ is 5-dimensional.

It cannot be spanned by $k < 5$ vectors.

MAT 281E – Homework 4

Due 03.12.2010

1. Let V be a k-dimensional subspace of $\mathbb{R}^n$. Show that $V^\perp$ is a subspace.

2. Does there exist a matrix whose row space contains $\begin{pmatrix} 1 & -2 & 1 \end{pmatrix}$ and whose null-space contains $\begin{pmatrix} -1 & 2 & 1 \end{pmatrix}$? If there exist such matrices, provide one. If not, explain why not.

3. In $\mathbb{R}^2$, describe two subspaces V_1, V_2 that are not orthogonal but such that any $x \in \mathbb{R}^2$ can be written as $x = x_1 + x_2$ where $x_1 \in V_1$ and $x_2 \in V_2$.

4. Let x, y be any two vectors. Show that

$$(x^T\, y)^2 \leq (x^T\, x)\,(y^T\, y). \tag{1}$$

   Hint : Consider $\left\| x - \frac{y^T\, x}{y^T\, y}\, y \right\|^2$.
   Note : This inequality is usually written as $\langle x, y \rangle \leq \|x\|\,\|y\|$, is very useful to know and is called _________ inequality.

5. Find the matrix that projects every point in $\mathbb{R}^3$ to the intersection of the planes $x + y + 2z = 0$ and $x + z = 0$.

6. Let P be the projection matrix that projects any vector onto a subspace V. What is the projection matrix for the subspace $V^\perp$? Please explain your answer.

7. (a) Let A be a $k \times k$ matrix whose rank is equal to k. If $A^2 = A$, show that actually $A = I$.
   (b) Let P be the projection matrix for a subspace V of $\mathbb{R}^n$. What is the condition on V such that P is invertible?

MAT 281E - Homework 4 Solutions

(1.) We need to show (i) for any $x, y \in V^{\perp}$, $x+y \in V^{\perp}$

(ii) for any $x \in V^{\perp}$, $\alpha x \in V^{\perp}$ for any real α.

(i) $x, y \in V^{\perp}$ means that $<x, v> = <y, v> = 0$ for any $v \in V$.

$\Rightarrow <x+y, v> = (x+y)^T v = x^T v + y^T v = <x, v> + <y, v> = 0$

$\Rightarrow x+y \in V^{\perp}$.

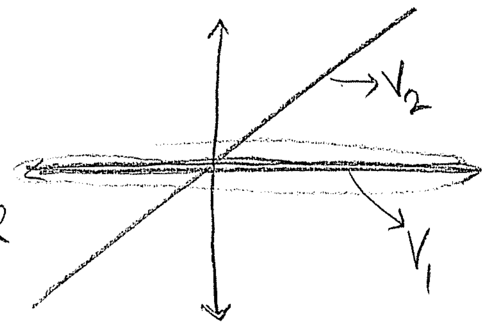
(ii) $x \in V^{\perp} \Rightarrow <x, v> = x^T v = 0$, where v can be any element of V.

Take an arbitrary α. $\Rightarrow <\alpha x, v> = \alpha x^T v = 0 \Rightarrow \alpha x \in V^{\perp}$.

(2.) $C(A^T)$ and $N(A)$ are orthogonal, but $\begin{bmatrix} 1 & -2 & 1 \end{bmatrix} \cdot \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix} \neq 0$

$\Rightarrow$ there is no such matrix.

(3.) Take $V_1 = \left\{ \alpha \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\}_{\alpha \in \mathbb{R}}$, $V_2 = \left\{ \alpha \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}_{\alpha \in \mathbb{R}}$



They are not orthogonal spaces but, we can write any vector in $\mathbb{R}^2$ $\begin{bmatrix} x \\ y \end{bmatrix}$ as, $\begin{bmatrix} x \\ y \end{bmatrix} = \alpha \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \gamma \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

By solving $\underbrace{\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}}_{A} \begin{bmatrix} \alpha \\ \gamma \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix}$, because A has 2 linearly indep. columns, so that it is invertible.

(4.) $0 \leq \left\| x - \frac{y^T x}{y^T y} y \right\|^2 = \left(x - \frac{y^T x}{y^T y} y \right)^T \left(x - \frac{y^T x}{y^T y} y \right) =$ "Schwarz Inequality"

$= x^T x - \frac{(x^T y)(y^T x)}{y^T y} - \frac{(y^T x)(y^T x)}{y^T y} + \frac{(y^T x)(y^T x)}{y^T y} \Rightarrow (x^T y)^2 \leq (x^T x)(y^T y)$

(6.) We know that for any x, the error vector $e=(x-Px)\in V^{\perp}$.
In fact e is the projection of x to $V^{\perp}$ because $(x-e)=Px \in V$.
$\Rightarrow$ The projection of any x to $V^{\perp}$ is $x-Px=(I-P)x$.
$(I-P)$ must be the projection matrix. $\left(\text{Notice that } \begin{array}{l}(I-P)^T=I-P \\ (I-P)^2=I-P\end{array}\right)$

(5.) We need to find the projection matrix to the null-space of

$A=\begin{bmatrix} 1 & 1 & 2 \\ 1 & 0 & 1 \end{bmatrix}$. We can do it in 2 ways.

(i) Instead of finding the projection matrix for $N(A)$, find the projection matrix P_1 for
$(N(A))^{\perp}=C(A^T)$ and take $I-P_1$ (using the result of Q6).

Let $V=A^T \Rightarrow P_1=V(V^TV)^{-1}V^T=\begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}/3$ $\quad P=I-P_1=\begin{bmatrix} 1 & 1 & -1 \\ 1 & 1 & -1 \\ -1 & -1 & 1 \end{bmatrix}/3$

computations are skipped!
But you should not!!!

(ii) $N(A)=\left\{\alpha\cdot\begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}\right\}_{\alpha\in\mathbb{R}} \Rightarrow \left\{\underbrace{\begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}}_{b_1}\right\}$ forms a basis for $N(A)$.

Compute the projection onto the column space of $B=[b_1]$.

$P=B(B^TB)^{-1}B^T=\begin{bmatrix} 1 & 1 & -1 \\ 1 & 1 & -1 \\ -1 & -1 & 1 \end{bmatrix}\cdot\left(\frac{1}{3}\right)$

(7.) (a) A is full-rank $\Rightarrow$ invertible. $A^{-1}(A^2)=A^{-1}A \Rightarrow A=I$.

(b) If P is invertible, since $P^2=P$ we have $P=I$.
Take an arbitrary $x\in\mathbb{R}^n \Rightarrow Px=x\in V \Rightarrow V\supset\mathbb{R}^n \Rightarrow V=\mathbb{R}^n$.

MAT 281E – Homework 5

Due 10.12.2010

1. (a) Find a vector x that minimizes $\|A\,x - b\|$ where

$$A = \begin{bmatrix} 1 & 0 & 2 \\ 2 & 1 & 5 \\ 0 & 2 & 2 \end{bmatrix}, \qquad b = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}.$$

   (b) Is the vector x you found in part (a) unique or can you find $\tilde{x} \neq x$ such that $\|A\,\tilde{x} - b\| = \|A\,x - b\|$? If x is not unique, provide such a $\tilde{x}$. If it is unique, explain why.

2. Consider two lines l_1, l_2, described by $l_1 = \big(x,\, 2x,\, x\big)$, $l_2 = \big(y,\, 3y,\, -1\big)$.

   (a) Find two points p, q where $p \in l_1$, $q \in l_2$ such that $\|p - q\|$ is minimized.

   (b) Are the points you found in part (a) unique – that is, can you find $\tilde{p} \in l_1$, $\tilde{q} \in l_2$ such that $\tilde{p} \neq p$ or $\tilde{q} \neq q$ but $\|\tilde{p} - \tilde{q}\| = \|p - q\|$? Please explain your answer.

3. If Q_1 and Q_2 are orthogonal matrices, show that $Q_1\,Q_2$ is also orthogonal.

4. We showed in class that if Q has orthonormal columns, then it preserves the lengths of vectors, i.e. $\|Q\,x\| = \|x\|$ for every x. Show that the converse is also true. That is, show that if $\|Q\,x\| = \|x\|$ for every x, then Q has orthonormal columns.

   Hint : Suppose that $Q = \begin{bmatrix} q_1 & q_2 & \dots & q_k \end{bmatrix}$ does not have orthonormal columns and construct an x such that $\|Q\,x\| \neq \|x\|$. (Why is this equivalent to what you are trying to show?)

5. Find the $Q\,R$ decomposition of

$$A = \begin{bmatrix} 1 & -1 & 0 & -3 \\ 1 & 1 & 2 & 1 \\ 1 & -1 & -2 & 1 \\ 1 & 1 & 0 & -3 \end{bmatrix}.$$

MAT 281 E - HW5 Solutions

1. (a) I don't know beforehand whether A has independent column or not, so I try $A^TAx = A^Tb$.

$$\Rightarrow A^TA = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 2 \\ 2 & 5 & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 & 2 \\ 2 & 1 & 5 \\ 0 & 2 & 2 \end{bmatrix} = \begin{bmatrix} 5 & 2 & 12 \\ 2 & 5 & 9 \\ 12 & 9 & 33 \end{bmatrix}, \quad A^Tb = \begin{bmatrix} 3 \\ 3 \\ 9 \end{bmatrix}$$

$$\left[\begin{array}{ccc|c} 5 & 2 & 12 & 3 \\ 2 & 5 & 9 & 3 \\ 12 & 9 & 33 & 9 \end{array}\right] \longrightarrow \left[\begin{array}{cccc} 5 & 2 & 12 & 3 \\ 0 & 21/5 & 21/5 & 9/5 \\ 0 & 21/5 & 21/5 & 9/5 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 5 & 2 & 12 & 3 \\ 0 & 21/5 & 21/5 & 9/5 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\Rightarrow \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 3/7 \\ 3/7 \\ 0 \end{bmatrix}$$ solves the system and minimizes $\|Ax-b\|$.

particular soln.

(To check: $(Ax-b)$ should be in $N(A^T)$.)

(b) x is not unique. There are other vectors that satisfy $A^TA\tilde{x} = A^Tb$ because the echelon form of A^TA has a zero-row (and therefore it is not full-rank.)

For instance, the special solution $y = \begin{bmatrix} -11/7 \\ -4/7 \\ 1 \end{bmatrix}$ can be added to x in (a) to find another vector. That is, for $\tilde{x} = x + y = \begin{bmatrix} -8/7 \\ -1/7 \\ 1 \end{bmatrix}$, $\|A\tilde{x}-b\| = \|Ax-b\|$

2. (a) We wish to minimize $\|e\|$

where $e = \begin{bmatrix} x \\ 2x \\ x \end{bmatrix} - \begin{bmatrix} y \\ 3y \\ -1 \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & -1 \\ 2 & -3 \\ 1 & 0 \end{bmatrix}}_{A} \begin{bmatrix} x \\ y \end{bmatrix} - \underbrace{\begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix}}_{b} \Rightarrow$ Equivalent to Minimizing $\|A\begin{bmatrix} x \\ y \end{bmatrix} - b\|$.

Columns of A are independent ($\Rightarrow (A^TA)$ is invertible.)

$A^TA\begin{bmatrix} x \\ y \end{bmatrix} = A^Tb \Rightarrow \begin{bmatrix} 6 & -7 \\ -7 & 10 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \end{bmatrix} \Rightarrow \left.\begin{matrix} 7x = 10y \\ 6x - 7 \cdot \frac{10}{7} x = -1 \end{matrix}\right\} \begin{matrix} x = 1/4 \\ y = \frac{7}{40} \end{matrix}$

Check: $(A\begin{bmatrix} x \\ y \end{bmatrix} - b) \in N(A^T)$ $\Rightarrow p = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} / 4 \quad q = \begin{bmatrix} 7/40 \\ 21/40 \\ -1 \end{bmatrix}$

(b) (A^TA) is invertible, so

$\begin{bmatrix} x \\ y \end{bmatrix}$ is unique $\Rightarrow p, q$ are unique.

3. $(Q_1 Q_2)^T (Q_1 Q_2) = Q_2^T Q_1^T Q_1 Q_2 = Q_2^T Q_2 = I \Rightarrow Q_1 Q_2$ is orth.

4. Suppose that $Q = [q_1\ q_2 \cdots q_k]$ does not have orthonormal columns.

Then, either (i) One of q_i's have $\|q_j\|^2 \neq 1$, or

(ii) $\|q_i\| = 1$ but $\langle q_j, q_\ell \rangle = q_j^T q_\ell \neq 0$ for some (j, ℓ) pair.

In either we can construct $x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_k \end{bmatrix}$ vectors such that $\|Qx\| \neq \|x\|$.

(i) Take $x_j = 1$ and $x_i = 0$ if $i \neq j$. Then $\|x\| = 1$, but $\|Qx\| = \|q_j\| \neq 1$.

(ii) Take $x_j = 1$, $x_\ell = 1$ and $x_i = 0$ if $i \neq j$ or $i \neq \ell$. Then $\|x\|^2 = 2$

But $Qx = q_j + q_\ell \Rightarrow \|Qx\|^2 = (q_j + q_\ell)^T (q_j + q_\ell) = q_j^T q_j + q_\ell^T q_\ell + 2 q_j^T q_\ell$

$= 2 + q_j^T q_\ell \neq 2 = \|x\|^2$.

MAT 281E – Homework 6 Solutions

1. True or False? (Notice the correction in (c).)

   (a) An $n \times n$ matrix always has n distinct eigenvalues. (F)

   (b) An $n \times n$ matrix always has n, possibly repeating, eigenvalues. (T)

   (c) An $n \times n$ matrix always has n eigenvectors that span $\mathbf{R}^n$. (F)

   (d) Every matrix has at least 1 eigenvector. (T)

   (e) If A and B have the same eigenvalues, they always have the same eigenvectors. (F)

   (f) If A and B have the same eigenvectors, they always have the same eigenvalues. (F)

   (g) If Q has 1/2 as an eigenvalue, then it cannot be orthogonal. (T)

   (h) If $A = S\,\Lambda\,S^{-1}$ where Λ is diagonal, then the rows of S have to be the eigenvectors of A. (F)

   (i) If $A = S\,\Lambda\,S^{-1}$ where Λ is diagonal, then the columns of S have to be the eigenvectors of A. (T)

   (j) An arbitrary matrix A can always be diagonalized as $A = S\,\Lambda\,S^{-1}$ where Λ is diagonal. (F)

2. Let A be an $n \times n$ matrix with all entries equal to 1 (i.e. $a_{i,j} = 1$). For $n = 2, 3$, find the eigenvalues and eigenvectors of A.

Here is one way to proceed :

For $n = 2$ we have,

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \end{bmatrix}.$$

We recognize the last expression as the scaled version of the projection matrix to the subspace S that contains (α, α). Thus, $\begin{bmatrix} 1 & 1 \end{bmatrix}^T$ must be an eigenvector. We note that $A \begin{bmatrix} 1 & 1 \end{bmatrix}^T = 2 \begin{bmatrix} 1 & 1 \end{bmatrix}^T$ so the eigenvalue is 2. The other eigenvector $\begin{bmatrix} 1 & -1 \end{bmatrix}^T$ comes from the orthogonal complement of S, with eigenvalue 0.

For $n = 3$ we have,

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \end{bmatrix}.$$

We recognize the last expression as the scaled version of the projection matrix to the subspace S that contains (α, α, α). Thus, $\begin{bmatrix} 1 & 1 & 1 \end{bmatrix}^T$ must be an eigenvector. We note that $A \begin{bmatrix} 1 & 1 & 1 \end{bmatrix}^T = 3 \begin{bmatrix} 1 & 1 & 1 \end{bmatrix}^T$ so the eigenvalue is 3. The other two eigenvectors $\begin{bmatrix} 1 & -1 & 0 \end{bmatrix}^T$, $\begin{bmatrix} 0 & 1 & -1 \end{bmatrix}^T$ come from the orthogonal complement of S, with eigenvalue 0.

Remark : Since A is symmetric, we know without computing anything that we can find n independent (even orthogonal if we like) eigenvectors. Once we do find n such eigenvectors, we can stop, since there can be no more. By the way, for $n = 3$, the eigenvectors (even if we require them to have unit energy) are not unique. Why not?

3. Suppose that A is a 3×3 matrix with eigenvalues λ_1, λ_2, λ_3 where the corresponding eigenvectors are x_1, x_2, x_3. What are the eigenvalues and eigenvectors of $2A - I$?

We have,

$$A\,x_1 = \lambda_1\,x_1, \quad A\,x_2 = \lambda_2\,x_2, \quad A\,x_1 = \lambda_3\,x_3.$$

Multiplying by 2 and subtracting multiples of x_i from both sides of the equations, we have,

$$2\,A\,x_1 - x_1 = (2\lambda_1 - 1)\,x_1, \quad 2\,A\,x_2 - x_2 = (2\lambda_2 - 1)\,x_2, \quad 2\,A\,x_3 - x_3 = (2\lambda_3 - 1)\,x_3.$$

Thus the eigenvalues are $(2\lambda_1 - 1)$, $(2\lambda_2 - 1)$, $(2\lambda_3 - 1)$ with associated eigenvectors x_1, x_2, x_3.

4. Find the eigenvalues of the following matrices.

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 3 & 0 \\ 4 & 5 & 6 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 2 & 3 & 0 & 0 & 0 \\ 0 & 0 & 4 & 5 & 6 \\ 0 & 0 & 0 & 7 & 8 \\ 0 & 0 & 0 & 0 & 9 \end{bmatrix}.$$

The eigenvalues are given by the diagonal entries. For A, these are 1, 3, 6. Check that $A - \lambda\,I$ is singular if λ is equal to an eigenvalue (these are all the eigenvalues because a 3×3 matrix cannot have more than 3 eigenvalues). Similarly, for B, the eigenvalues are 1, 3, 4, 7, 9.

5. Let $y(n) = 2\,y(n-1) + 3\,y(n-2)$. Suppose that $y(1) = 4$, $y(0) = 0$. Compute $y(101)$.

Define

$$u_n = \begin{bmatrix} y(n) \\ y(n-1) \end{bmatrix}.$$

Then we have,

$$u_n = \underbrace{\begin{bmatrix} 2 & 3 \\ 1 & 0 \end{bmatrix}}_{A} u_{n-1}.$$

We can now write

$$u_{101} = \begin{bmatrix} y(101) \\ y(100) \end{bmatrix} = A^{100} \begin{bmatrix} y(1) \\ y(0) \end{bmatrix} = A^{100} \begin{bmatrix} 4 \\ 0 \end{bmatrix}.$$

Let us diagonalize A to compute A^{100}. To find the eigenvalues, we compute the roots of $\det(A - \lambda\,I)$. That is,

$$\begin{vmatrix} 2-\lambda & 3 \\ 1 & -\lambda \end{vmatrix} = \lambda^2 - 2\,\lambda - 3 = (\lambda - 3)\,(\lambda + 1).$$

So the eigenvalues are 3 and -1.

To compute the eigenvector for $\lambda = 3$, we look at the nullspace of $A - 3I$:

$$A - 3I = \begin{bmatrix} -1 & 3 \\ 1 & -3 \end{bmatrix}$$

From this we see that $(3, 1)$ is an eigenvector for $\lambda = 3$.

To compute the eigenvector for $\lambda = -1$, we look at the nullspace of $A + I$:

$$A + I = \begin{bmatrix} 3 & 3 \\ 1 & 1 \end{bmatrix}$$

From this we see that $(1, -1)$ is an eigenvector for $\lambda = -1$.

Thus, we can write A as,

$$A \underbrace{\begin{bmatrix} 3 & 1 \\ 1 & -1 \end{bmatrix}}_{S} = \underbrace{\begin{bmatrix} 3 & 0 \\ 0 & -1 \end{bmatrix}}_{\Lambda} S.$$

We see that

$$S \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ 0 \end{bmatrix} = \begin{bmatrix} y(1) \\ y(0) \end{bmatrix}$$

or

$$\begin{bmatrix} 1 \\ 1 \end{bmatrix} = S^{-1} \begin{bmatrix} y(1) \\ y(0) \end{bmatrix}$$

Since $A^{100} = S\,\Lambda^{100}\,S^{-1}$, we obtain

$$\begin{bmatrix} y(101) \\ y(100) \end{bmatrix} = \begin{bmatrix} 3 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 3^{100} & 0 \\ 0 & (-1)^{100} \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 3^{101} + 1 \\ 3^{100} - 1 \end{bmatrix}.$$

MAT 281E – Homework 7

Due 11.01.2011

1. Construct a 3×3 matrix whose column space contains $\begin{pmatrix} 1 & 1 & 1 \end{pmatrix}$ and $\begin{pmatrix} 1 & 1 & 0 \end{pmatrix}$ but not $\begin{pmatrix} 1 & 0 & 1 \end{pmatrix}$.

2. Consider the line l described as the intersection of the planes $x+y+z = 0$ and $x+2y+z = 0$. Construct, if you can, a 3×3 matrix A where $C(A) = l$.

3. Consider the line $l = \begin{pmatrix} \alpha & \alpha - 1 & 2\,\alpha \end{pmatrix}$. Construct, if you can, a 3×3 matrix A where $C(A) = l$.

4. Let $A = E_1\, R$ and $B = E_2\, R$ where E_1 and E_2 are invertible. We do not have further information about R. Below are four questions regarding the four fundamental subspaces. If you think that the information is not sufficient to answer the questions, write so.

   (a) Can you find a relation between $C(A)$ and $C(B)$?

   (b) Can you find a relation between $C(A^T)$ and $C(B^T)$?

   (c) Can you find a relation between $N(A)$ and $N(B)$?

   (d) Can you find a relation between $N(A^T)$ and $N(B^T)$?

5. (This was the last question in HW5) Find the $Q\,R$ decomposition of

$$A = \begin{bmatrix} 1 & -1 & 0 & -3 \\ 1 & 1 & 2 & 1 \\ 1 & -1 & -2 & 1 \\ 1 & 1 & 0 & -3 \end{bmatrix}.$$

6. Let λ_1, λ_2, λ_3, be the distinct non-zero eigenvalues of a 3×3 matrix B, where the associated eigenvectors are x_1, x_2, x_3. What are the eigenvalues and eigenvectors of B^{-1}?

7. Consider the plane P_1 in $\mathbb{R}^4$ described by $x_1+x_2-x_3 = 2$ and the line $l = \begin{pmatrix} \alpha, & \alpha+1, & -2\alpha, -\alpha \end{pmatrix}$. Find the points $p \in P_1$, $q \in l$ that minimize $\|p - q\|$. Are these points unique?

8. Let A be a 17×17 matrix where $A_{ij} = i - j$. Notice that $A^T = -A$. Let $x = \begin{bmatrix} 1 & 2 & \dots & 17 \end{bmatrix}^T$. What is $x^T\, A\, x$?

9. Let B be a 3×3 matrix and suppose that the eigenvectors x_1, x_2, x_3, with associated eigenvectors λ_1, λ_1, λ_2, span $\mathbb{R}^3$. Consider the matrix

$$A = \begin{bmatrix} B & \mathbf{0} \\ \mathbf{0} & 1 \end{bmatrix}.$$

Find four vectors y_1, y_2, y_3 and y_4 that span $\mathbb{R}^4$ and are also eigenvectors of A.

10. Consider the matrix

$$A = \begin{bmatrix} 0 & 2 & 0 & 0 \\ 2 & 0 & 0 & 0 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 2 & 1 \end{bmatrix}.$$

Find a decomposition of A as $A = Q\Lambda\, Q^T$ where Q is orthogonal and Λ is diagonal.

MAT281E — HW7 Solutions

(1.) $(1\ 0\ 1)$ cannot be written as a linear combination of $(1\ 1\ 1)$ and $(1\ 1\ 0)$ so, $A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$ does it.

(2.) ℓ is the nullspace of $B = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 1 \end{bmatrix}$.

To find the nullspace: $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$

$\Rightarrow \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$ is in the null-space of B (it also spans $N(B)$ since $N(B)$ 1-dimensional)

$\Rightarrow A = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \begin{bmatrix} 1 & a & b \end{bmatrix}$ for arbitrary a, b is a matrix with $C(A) = \ell$.

(3.) The line is not a subspace because it doesn't pass through the origin. We cannot find A with $C(A) = \ell$, since $C(A)$ has to be a subspace.

(4.) (a) We can only say that their dimension will be the same.

(c) If $Ax = 0 \Rightarrow Rx = E_1^{-1} Ax = 0 \Rightarrow Bx = E_2 Rx = 0 \Rightarrow N(A) \subset N(B)$

If $Bx = 0 \Rightarrow Ax = 0$ similarly $\Rightarrow N(B) \subset N(A) \longrightarrow N(A) = N(B)$

(b) $C(A^T) = N(A)^{\perp} = N(B)^{\perp} = C(B^T)$.

(d) $\dim N(A^T) = \dim N(B^T)$. No further conclusion from the information given.

5. $A = \begin{bmatrix} c_1 & c_2 & c_3 & c_4 \end{bmatrix}$

$$q_1 = \frac{c_1}{\sqrt{\langle c_1, c_1 \rangle}} = \frac{c_1}{2} = \begin{bmatrix} 1/2 \\ 1/2 \\ 1/2 \\ 1/2 \end{bmatrix}$$

$$\tilde{q}_2 = c_2 - \underbrace{\langle c_2, q_1 \rangle}_{=0} q_1 = c_2 \; ; \quad q_2 = \frac{\tilde{q}_2}{\sqrt{\langle q_2, q_2 \rangle}} = \frac{c_2}{2} = \begin{bmatrix} -1/2 \\ 1/2 \\ -1/2 \\ 1/2 \end{bmatrix}$$

$$\tilde{q}_3 = c_3 - \langle c_3, q_1 \rangle q_1 - \langle c_3, q_2 \rangle q_2 = \begin{bmatrix} 0 \\ 2 \\ -2 \\ 0 \end{bmatrix} - 0 \cdot q_1 - 2 \cdot q_2 = \begin{bmatrix} 1 \\ 1 \\ -1 \\ -1 \end{bmatrix}$$

$$q_3 = \frac{\tilde{q}_3}{\sqrt{\langle \tilde{q}_3, \tilde{q}_3 \rangle}} = \tilde{q}_3 / 2 = \begin{bmatrix} 1/2 \\ 1/2 \\ -1/2 \\ -1/2 \end{bmatrix}$$

$$\tilde{q}_4 = c_4 - \langle c_4, q_1 \rangle q_1 - \langle c_4, q_2 \rangle q_2 - \langle c_4, q_3 \rangle q_3$$

$$= \begin{bmatrix} -3 \\ 1 \\ 1 \\ -3 \end{bmatrix} - (-2) \cdot q_1 - 0 \cdot q_2 - 0 \cdot q_3 = \begin{bmatrix} -2 \\ 2 \\ 2 \\ -2 \end{bmatrix}$$

$$q_4 = \frac{\tilde{q}_4}{\sqrt{\langle \tilde{q}_4, \tilde{q}_4 \rangle}} = \tilde{q}_4 / 4$$

$$= \begin{bmatrix} -1/2 & 1/2 & 1/2 & -1/2 \end{bmatrix}^T$$

Now $c_1 = 2q_1$; $c_2 = 2q_2$; $c_3 = 2q_3 + 2q_2$

$c_4 = 4q_4 - 2q_1$

$$\Rightarrow A = \underbrace{\begin{bmatrix} q_1 & q_2 & q_3 & q_4 \end{bmatrix}}_{Q} \overbrace{\begin{bmatrix} 2 & 0 & 0 & -2 \\ 0 & 2 & 2 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 4 \end{bmatrix}}^{R}$$

(6.) $Bx_i = \lambda_i x_i$ for $i = 1, 2, 3$.

$\Rightarrow \frac{1}{\lambda_i} x_i = B^{-1} x_i \Rightarrow$ eigvectors: x_1, x_2, x_3

eigvalues: $\frac{1}{\lambda_1}, \frac{1}{\lambda_2}, \frac{1}{\lambda_3}$ (Notice $\lambda_i \neq 0$ since B is invertible)

(7.) P_1 is the solution set of $\underbrace{\begin{bmatrix} 1 & 1 & -1 & 0 \end{bmatrix}}_{C} \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}}_{x} = 2$

The soln. set is described as

$y_p + y_{s_1} \cdot \alpha_1 + y_{s_2} \cdot \alpha_2 + y_{s_3} \cdot \alpha_3$ where y_{s_i}'s are special solutions, y_p is a particular soln. and α_i's are scalars.

Free variables: x_2, x_3, x_4

Pivot var: x_1.

$\Rightarrow y_p = \begin{bmatrix} 2 \\ 0 \\ 0 \\ 0 \end{bmatrix}$ (Set free var. to zero & solve).

$y_{s_1} = \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix}$ (Set $x_3 = x_4 = 0$, $x_2 = 1$ and solve $Cx = 0$)

$y_{s_2} = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}$, $y_{s_3} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$ similarly.

$\Rightarrow$ A pt. on the plane is given by $\underbrace{\begin{bmatrix} -1 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{D} \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{bmatrix} + \underbrace{\begin{bmatrix} 2 \\ 0 \\ 0 \\ 0 \end{bmatrix}}_{e}$.

$$p-q = D\begin{bmatrix}\alpha_1\\ \alpha_2\\ \alpha_3\end{bmatrix} + e - \left(\begin{bmatrix}1\\1\\-2\\-1\end{bmatrix}\alpha_4 + \begin{bmatrix}0\\1\\0\\0\end{bmatrix}\right) = \underbrace{\begin{bmatrix}1 & 1 & 0 & 1\\ 1 & 0 & 0 & 1\\ 0 & 1 & 0 & -2\\ 0 & 0 & 1 & -1\end{bmatrix}}_{A}\underbrace{\begin{bmatrix}\alpha_1\\ \alpha_2\\ \alpha_3\\ -\alpha_4\end{bmatrix}}_{x} - \underbrace{\begin{bmatrix}-2\\1\\0\\0\end{bmatrix}}_{b} = e$$

Solve $A^TAx = A^Tb.$

$$A^TA = \begin{bmatrix}2 & 1 & 0 & 2\\ 1 & 2 & 0 & -1\\ 0 & 0 & 1 & -1\\ 2 & -1 & -1 & 7\end{bmatrix}, \quad A^Tb = \begin{bmatrix}-1\\-2\\0\\-1\end{bmatrix}$$

$$\left[\begin{array}{cccc|c}2 & 1 & 0 & 2 & -1\\ 1 & 2 & 0 & -1 & -2\\ 0 & 0 & 1 & -1 & 0\\ 2 & -1 & -1 & 7 & -1\end{array}\right] \longrightarrow \begin{bmatrix}2 & 1 & 0 & 2 & -1\\ 0 & 3/2 & 0 & -2 & -3/2\\ 0 & 0 & 1 & -1 & 0\\ 0 & -2 & -1 & 5 & 0\end{bmatrix} \longrightarrow \begin{bmatrix}2 & 1 & 0 & 2 & -1\\ 0 & 1 & 0 & -4/3 & -1\\ 0 & 0 & 1 & -1 & 0\\ 0 & 0 & -1 & 7/3 & -2\end{bmatrix}$$

$$\longrightarrow \left[\begin{array}{cccc|c}1 & 1/2 & 0 & 1 & -1/2\\ 0 & 1 & 0 & -4/3 & -1\\ 0 & 0 & 1 & -1 & 0\\ 0 & 0 & 0 & 4/3 & -2\end{array}\right] \Longrightarrow \begin{aligned} -\alpha_4 &= -3/2\\ \alpha_3 &= -3/2\\ \alpha_2 &= -1 + 4/3\,\alpha_4 = -3\\ \alpha_1 &= \frac{-1}{2} - \alpha_4 - \frac{\alpha_2}{2} = 5/2\end{aligned}$$

$$\Rightarrow p = D\begin{bmatrix}\alpha_1\\ \alpha_2\\ \alpha_3\end{bmatrix} + \begin{bmatrix}2\\0\\0\\0\end{bmatrix} = \begin{bmatrix}3/2\\5/2\\-3\\-3/2\end{bmatrix}, \quad q = \begin{bmatrix}1\\1\\-2\\-1\end{bmatrix}\alpha_4 + \begin{bmatrix}0\\1\\0\\0\end{bmatrix} = \begin{bmatrix}3/2\\5/2\\-3\\-3/2\end{bmatrix} = p$$

p & q are unique because A^TA is invertible (4 pivots).

(8.) $x^T A x = x^T (Ax) = x^T c = c^T x$

$x^T A x = (x^T A) x = (A^T x)^T x = (-Ax)^T x = -c^T x$

$\Rightarrow x^T A x = -x^T A x \Rightarrow 2(x^T A x) = 0$

(9.) $y_1 = \begin{bmatrix} x_1 \\ 0 \end{bmatrix}, \quad y_2 = \begin{bmatrix} x_2 \\ 0 \end{bmatrix}, \quad y_3 = \begin{bmatrix} x_3 \\ 0 \end{bmatrix}, \quad y_4 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$

$\Rightarrow A y_1 = \begin{bmatrix} Bx_1 \\ 1 \cdot 0 \end{bmatrix} = \lambda_1 \begin{bmatrix} x_1 \\ 0 \end{bmatrix} = \lambda_1 y_1$

Similarly $Ay_2 = \lambda_2 y_2$, $Ay_3 = \lambda_3 y_3$

and $A y_4 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} = y_4$

y_1, y_2, y_3, y_4 span $\mathbb{R}^4$ (Why?)

(10.) $A = \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix}$ where $A_1 = \begin{bmatrix} 0 & 2 \\ 2 & 0 \end{bmatrix}$, $A_2 = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$

Eigv of A_1 are the solutions of $\lambda^2 - 4 = 0 \Rightarrow \lambda_1 = 2, \lambda_2 = -2$

$A_1 - 2I = \begin{bmatrix} -2 & 2 \\ 2 & -2 \end{bmatrix} \Rightarrow$ associated eigvector of A_1 $= \begin{bmatrix} 1 \\ 1 \end{bmatrix} = c_1$

$A_1 + 2I = \begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix} \Rightarrow$ associated eigvector of A_1 $= \begin{bmatrix} 1 \\ -1 \end{bmatrix} = c_2$

Notice that

$A \underbrace{\begin{bmatrix} c_1 \\ 0 \\ 0 \end{bmatrix}}_{e_1} = \begin{bmatrix} A_1 c_1 \\ 0 \end{bmatrix} = 2 \underbrace{\begin{bmatrix} c_1 \\ 0 \\ 0 \end{bmatrix}}_{e_1}$

Similarly $e_2 = \begin{bmatrix} c_2 \\ 0 \\ 0 \end{bmatrix}$ is an eigvector with eigvalue $= -2$.

$\Rightarrow e_1$ is an eigenvector of A with eigenvalue $= 2$.

Eigvectors of A_2 are the solutions of $\det(A_2 - \lambda I) = 0$

$$\Rightarrow (1-\lambda)^2 - 4 = \lambda^2 - 2\lambda - 3 = (\lambda - 3)(\lambda + 1) = 0 \Rightarrow \lambda_3 = 3, \ \lambda_4 = -1$$

$$A_2 - 3I = \begin{bmatrix} -2 & 2 \\ 2 & -2 \end{bmatrix} \Rightarrow \text{associated eigvector of } A_2 = \underbrace{\begin{bmatrix} 1 \\ 1 \end{bmatrix}}_{c_3}$$

$$A_2 + I = \begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix} \Rightarrow \text{associated eigvector of } A_2 = \underbrace{\begin{bmatrix} 1 \\ -1 \end{bmatrix}}_{c_4}$$

$$\Rightarrow e_3 = \begin{bmatrix} 0 \\ 0 \\ c_3 \end{bmatrix}, \quad e_4 = \begin{bmatrix} 0 \\ 0 \\ c_4 \end{bmatrix}$$ are eigvectors of A with eigenvalue 3 and -1.

$$\Rightarrow A \underbrace{\begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} & 0 & 0 \\ 1/\sqrt{2} & -1/\sqrt{2} & 0 & 0 \\ 0 & 0 & 1/\sqrt{2} & 1/\sqrt{2} \\ 0 & 0 & 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix}}_{Q} = Q \underbrace{\begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}}_{\Lambda}$$

$$\Rightarrow A = Q \Lambda Q^T$$

Remark: We can work with submatrices if A is block-diagonal.

MAT 281E – Linear Algebra and Applications

Midterm Examination I

05.11.2010

(20 pts) 1. (a) Find the matrix X that satisfies the equation $A\,X = B$ where,

$$A = \begin{bmatrix} -2 & 3 & 0 \\ 4 & -6 & -1 \\ 6 & -3 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 2 & 3 & 5 \\ -3 & -7 & -11 \\ -6 & -3 & -9 \end{bmatrix}$$

(b) Find the matrix Y that satisfies the equation $Y\,C = D$ where,

$$C = \begin{bmatrix} -2 & 4 & 6 \\ 3 & -6 & -3 \\ 0 & -1 & 0 \end{bmatrix}, \quad D = \begin{bmatrix} 2 & -3 & -6 \\ 3 & -7 & -3 \\ 5 & -11 & -9 \end{bmatrix}.$$

(Hint for part (b) : Take a good look at the matrices in both parts. Also notice that you are <u>not</u> asked to solve $C\,Y = D$.)

(20 pts) 2. Find the LU decomposition of

$$A = \begin{bmatrix} -4 & 0 & -2 \\ 0 & 2 & 3 \\ 16 & -4 & 1 \end{bmatrix}.$$

(30 pts) 3. Consider the system of equations

$$\underbrace{\begin{bmatrix} 1 & 1 & -2 & 1 & 0 \\ 2 & 2 & -4 & 1 & -1 \\ 1 & 1 & -1 & 2 & 0 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix}}_{x} = \underbrace{\begin{bmatrix} 4 \\ 7 \\ 4 \end{bmatrix}}_{b}.$$

(a) Describe $N(A)$, the nullspace of A.

(b) What is the rank of A?

(c) What is the dimension of $N(A)$?

(d) Describe the solution set of $A\,x = b$.

(15 pts) 4. (a) Let C be the set of vectors of the form $\begin{bmatrix} x \\ y \end{bmatrix}$ where $x \geq 0$, $y \geq 0$. Is C a subspace of $\mathbb{R}^2$? Please explain your answer.

(b) Is it possible to find a 3×2, non-zero matrix A such that, the set of vectors of the form '$A \begin{bmatrix} x \\ y \end{bmatrix}$', where $x \geq 0$, $y \geq 0$, form a subspace of $\mathbb{R}^3$? If it is possible, provide such a matrix. If you think it is not possible, explain why not.

(15 pts) 5. True or False? The following subsets of $\mathbb{R}^3$ are also subspaces of $\mathbb{R}^3$.

(a) All vectors $\begin{pmatrix} x_1 & x_2 & x_3 \end{pmatrix}$ with $x_2 = 1$.

(b) The vector $\begin{pmatrix} 1 & 1 & 1 \end{pmatrix}$ alone.

(c) All vectors $\begin{pmatrix} x_1 & x_2 & x_3 \end{pmatrix}$ with $x_2 - 2x_3 = x_1$.

(d) All vectors $\begin{pmatrix} x_1 & x_2 & x_3 \end{pmatrix}$ with $x_3 = x_2/x_1$.

(e) All vectors $\begin{pmatrix} x_1 & x_2 & x_3 \end{pmatrix}$ with $x_2^2 - x_3\,x_2 = 0$.

(f) All vectors of the form '$\alpha \begin{pmatrix} 1 & 2 & 3 \end{pmatrix}$' where $\alpha \geq 0$.

(g) All vectors of the form '$\alpha \begin{pmatrix} 1 & 2 & 3 \end{pmatrix}$' where $-1 \leq \alpha \leq 1$.

(h) All vectors $\begin{pmatrix} x_1 & x_2 & x_3 \end{pmatrix}$ with $x_1 = x_2$.

(i) All vectors $\begin{pmatrix} x_1 & x_2 & x_3 \end{pmatrix}$ with $x_1^2 = x_2^2$.

(j) All vectors $\begin{pmatrix} x_1 & x_2 & x_3 \end{pmatrix}$ with $x_1^3 = x_2^3$.

MAT 281E – Linear Algebra and Applications

Midterm Examination II

17.12.2010

Student Name : ____________________

Student Num. : ____________________

5 Questions, 120 Minutes

Please Show Your Work!

(10 pts) 1. Consider the space S, spanned by

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \qquad \mathbf{v}_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}.$$

(a) Construct a matrix A such that $C(A) = S$ (here $C(A)$: the column space of A).

(b) Find a vector from the orthogonal complement of S.

(20 pts) 2. Suppose that A is a 3×3 matrix, whose rank is 2 (i.e. it has 2 independent columns) and

$$\mathbf{v}_1^T A = \begin{bmatrix} 0 & 2 & 0 \end{bmatrix},$$
$$\mathbf{v}_2^T A = \begin{bmatrix} 0 & 1 & 0 \end{bmatrix},$$

where

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix}, \qquad \mathbf{v}_2 = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}.$$

(a) What is the dimension of $N(A^T)$, the left nullspace of A?

(b) Find a basis for $N(A^T)$.

(c) Find the matrix P that projects any point to $N(A^T)$.

(d) Find the matrix Q that projects any point to $C(A)$, the column space of A.

(15 pts) 3. Consider the lines $l_1 = \Big(x,\, 2x,\, x+3,\, -x\Big)$, $l_2 = \Big(1-y,\, -2y,\, -1-y,\, 2\Big)$ in $\mathbb{R}^4$. Find two points $p \in l_1$, $q \in l_2$ that minimize $\|p - q\|$.

(25 pts) 4. Let V be a subspace in $\mathbb{R}^3$ spanned by

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \qquad \mathbf{v}_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

and l, a line described as $l = \Big(x,\, 1,\, -x\Big)$.

(a) Find two points $p \in V$, $q \in l$ that minimize $\|p - q\|$.

(b) Find two more points $\tilde{p} \in V$, $\tilde{q} \in l$, such that $\tilde{p} \neq p$, $\tilde{q} \neq q$ and $\|p - q\| = \|\tilde{p} - \tilde{q}\|$.

(30 pts) 5. (a) Suppose we are given

$$\mathbf{a}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \qquad \mathbf{a}_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \qquad \mathbf{a}_3 = \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix}$$

that span $\mathbb{R}^3$.

Let $\mathbf{q}_1 = \alpha\, \mathbf{a}_1$ where α is a scalar. Select α and find two more vectors $\mathbf{q}_2$, $\mathbf{q}_3$, using the Gram-Schmidt procedure, such that $\{\mathbf{q}_1, \mathbf{q}_2, \mathbf{q}_3\}$ is an orthonormal basis for $\mathbb{R}^3$.

(b) Consider the plane P described by the equation $x + y + z = 3$. Find the closest point of P to $(1, 2, 3)$.

MAT 281E – Linear Algebra and Applications

Final Examination

18.01.2011

Student Name : ____________________

Student Num. : ____________________

5 Questions, 120 Minutes

Please Show Your Work!

(20 pts) 1. Consider the system of equations

$$\underbrace{\begin{bmatrix} 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & -3 \\ 1 & 3 & -5 & -3 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}}_{x} = \underbrace{\begin{bmatrix} 0 \\ 2 \\ -6 \end{bmatrix}}_{b}.$$

(a) Describe the solution set of $A\,x = b$.

(b) What is the rank of A? What are the dimensions of the four fundamental subspaces, $N(A)$, $C(A)$, $N(A^T)$, $C(A^T)$?

(15 pts) 2. Consider the plane P in $\mathbb{R}^3$ described by the equation $x + y + 2z = 0$.

(a) Find two vectors $\mathbf{v}_1$, $\mathbf{v}_2$, that span P.

(b) Find a 3×3 matrix A such that $N(A) = P$.

(c) Find a 3×3 matrix B such that $C(B) = P$.

(20 pts) 3. Let P be a plane in $\mathbb{R}^3$. Suppose we are given three points on P as,

$$p_1 = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}, \quad p_2 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \quad p_3 = \begin{bmatrix} 2 \\ 2 \\ 3 \end{bmatrix}.$$

Let A, b be such that the solution set of $A\,x = b$ is P.

(a) What is the dimension of $N(A)$?

(b) Find a basis for $N(A)$.

(c) Let

$$q = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}.$$

Find $p \in P$ that minimizes $\|p - q\|$.

(20 pts) 4. Let

$$A = \begin{bmatrix} 1 & -3 \\ -3 & 1 \end{bmatrix}.$$

(a) Find the eigenvalues and eigenvectors of A.

(b) Find an orthogonal Q and a diagonal Λ such that $A = Q\,\Lambda\,Q^T$.

(c) Compute A^{20}.

(25 pts) 5. Suppose we are given

$$\mathbf{a}_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \qquad \mathbf{a}_2 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}, \qquad \mathbf{a}_3 = \begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix}$$

that span $\mathbb{R}^3$.

Also, let

$$A = \underbrace{\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}}_{\mathbf{u}} \underbrace{\begin{bmatrix} 1 & 1 & 0 \end{bmatrix}}_{\mathbf{a}_1^T}.$$

(a) Apply the Gram-Schmidt procedure to the vectors $\{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3\}$ to find three vectors $\{\mathbf{q}_1, \mathbf{q}_2, \mathbf{q}_3\}$ which form an orthonormal basis for $\mathbb{R}^3$.

(b) What are the dimensions of $N(A)$ and $C(A)$?

(c) Find three eigenvectors, $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$, of A that span $\mathbb{R}^3$. What are the associated eigenvalues?