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ON THE FREQUENCY RESPONSE FUNCTION OF A VISCOUSLY DAMPED CANTILEVER SIMPLY SUPPORTED IN-SPAN

Haluk Erol
Faculty of Mechanical Engineering,
Technical University of Istanbul,
80191 Gümüşsuyu, Istanbul, Turkey.
Email: erolha@itu.edu.tr

Metin Gürgöze
Faculty of Mechanical Engineering,
Technical University of Istanbul,
80191 Gümüşsuyu, Istanbul, Turkey.
Email: gurgozem@itu.edu.tr

ABSTRACT

This paper deals with the determination of the frequency response function of a cantilevered Bernoulli-Euler beam which is viscously damped by a single damper. The beam is simply supported in-span. The frequency response function is obtained through a formula that was established for the receptance matrix of discrete linear systems subjected to linear constraint equations, by considering the simple support as a linear constraint imposed on generalized co-ordinates. The comparison of the numerical results obtained via a boundary value problem formulation justifies the approach used here.

INTRODUCTION

The second author recently established a formula for the receptance matrix of viscously damped discrete systems subjected to several constraint equations in reference [1]. The reliability of the formula derived was tested on an academic example of a spring-mass system with three degrees of freedom the coordinates of which were assumed to be subjected to a constraint equation. In order to put forward the applicability of the method better, the formula was applied in reference [2] to a more complex but practical system. The system was made up of a cantilevered beam simply supported at a given distance from the fixed end. It was desired to determine the amplitude distribution of the beam due to harmonically varying vertical force acting at a given point. The problem could be posed as to find the frequency response function of the beam. The present study deals with the same system as in reference [2], the difference being, that here also a viscous damping of the beam is allowed by a single viscous damper.

THEORY

The problem can best be stated referring to the cantilevered beam shown in Figure 1. The Bernoulli-Euler beam, viscously damped by a viscous damper of damping constant c at $x = \alpha L$ is assumed to be simply supported at a distance $s^* = \eta L$ from the fixed end. At the distance $x = \gamma L$, a harmonically varying force $F(t)$ is acting on the beam. Now it is desired to determine the amplitude distribution of the beam due to this force. This problem can be posed also as to find the frequency response function of the beam.

Application Of The Formula In Reference [1]

Let us begin with the mechanical system in Figure 1 where it is assumed first that the support does not exist. The equation of the motion of the beam is [3]

$$EIw^{IV}(x,t) + m\ddot{w}(x,t) + c\dot{w}(x,t)\,\delta(x-\alpha L) = F(t)\,\delta(x-\gamma L) \tag{1}$$

the exciting force being

$$F(t) = F_0 e^{i\Omega t} \ , \tag{2}$$

where the primes and overdots denote partial derivatives with respect to x and time t respectively, and i is the imaginary unit. EI is the bending rigidity and m is mass per unit length of the beam. $\delta(x)$ denotes the Dirac function and c denotes the viscous damping coefficient.

The corresponding boundary conditions are

$$w(0,t) = w'(0,t) = w''(L,t) = w'''(L,t) = 0\,. \tag{3}$$

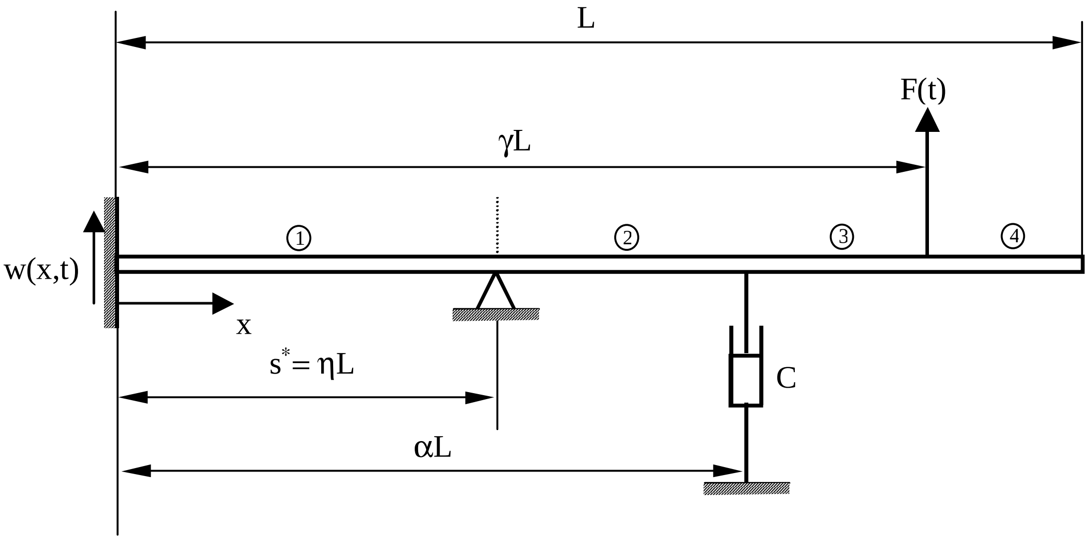

Figure 1. Viscously damped cantilevered beam simply supported in-span, subject to a harmonically varying force.

An approximate series solution of the differential equation (1) can be taken in the form

$$w(x,t) \approx \sum_{r=1}^{n} w_r(x)\,\eta_r(t) \qquad (4)$$

where the $w_r(x)$ are the orthogonal eigenfunctions of the bare clamped-free beam, normalised with respect to the mass density and $\eta_r(t)$ are the generalized co-ordinates. After substitution of expression (4) into the differential equation (1) and application of the Galerkin procedure, the system of the modal equations, i.e., the system of differential equations for the $\eta_i(t)$, is obtained as in reference [3]

$$\ddot{\eta}_i(t) + c w_i(\alpha L) \sum_{j=1}^{n} w_j(\alpha L)\,\dot{\eta}_j(t) + \omega_i^2 \eta_i(t) = N_i(t), \quad (i = 1,\ldots,n) \qquad (5)$$

where

$$\omega_i^2 = (\beta_i L)^4 \frac{EI}{mL^4}, \quad \bar{\beta}_1 = \beta_1 L = 1.875104068712,$$

$$\bar{\beta}_2 = \beta_2 L = 4.694091132974, \ldots,$$

$$N_i(t) = F(t)\, w_i(\gamma L). \qquad (6)$$

The system of differential equations in equation (5) can be written in matrix notation as

$$\ddot{\boldsymbol{\eta}}(t) + \mathbf{D}\,\dot{\boldsymbol{\eta}}(t) + \boldsymbol{\omega}^2 \boldsymbol{\eta}(t) = \mathbf{N}(t) \qquad (7)$$

where

$$\boldsymbol{\eta}(t) = [\eta_1(t) \; \ldots \; \eta_n(t)]^T, \quad \boldsymbol{\omega}^2 = \mathbf{diag}\left(\omega_i^2\right), \quad \mathbf{N}(t) = \bar{\mathbf{N}}\, e^{i\Omega t},$$

$$\bar{\mathbf{N}} = F_0\, \mathbf{w}(\gamma L), \quad \mathbf{D} = c\mathbf{w}(\alpha L)\mathbf{w}^T(\alpha L),$$

$$\mathbf{w}(x) = [w_1(x) \; \ldots \; w_n(x)]^T. \qquad (8)$$

ω_i (i = 1, …, n) are the eigenfrequencies of the bare cantilever beam.

Substitution of

$$\boldsymbol{\eta}(t) = \bar{\boldsymbol{\eta}}\, e^{i\Omega t} \qquad (9)$$

into the matrix differential equation (7) yields

$$\bar{\boldsymbol{\eta}} = \mathbf{H}(\Omega)\,\bar{\mathbf{N}} \qquad (10)$$

where the receptance matrix is in the form

$$\mathbf{H}(\Omega) = \left(-\Omega^2 \mathbf{I} + i\Omega\mathbf{D} + \boldsymbol{\omega}^2\right)^{-1}. \qquad (11)$$

Let us now return to the actual system with the support at x = ηL. The introduction of the support leads to the constraint equation

$$\sum_{r=1}^{n} w_r(s^*)\,\eta_r(t) = 0 \tag{12}$$

which can be written compactly as

$$\mathbf{a}_1^T\,\boldsymbol{\eta} = 0 \tag{13}$$

where

$$\mathbf{a}_1^T = \mathbf{w}^T(s^*) = \left[w_1(s^*)\ \ldots\ w_n(s^*)\right]^T,\ \ s^* = \eta L. \tag{14}$$

The amplitude vector $\bar{\boldsymbol{\eta}}$ in the constrained case can be written from equation (10) analogously as

$$\bar{\boldsymbol{\eta}} = \mathbf{H}_{cons}(\Omega)\,\overline{\mathbf{N}} \tag{15}$$

where from reference [1] the receptance matrix of the constrained system reads

$$\mathbf{H}_{cons}(\Omega) = \mathbf{H}(\Omega)\left[\mathbf{I} - \frac{\mathbf{w}(s^*)\,\mathbf{w}^T(s^*)\,\mathbf{H}(\Omega)}{\mathbf{w}^T(s^*)\,\mathbf{H}(\Omega)\,\mathbf{w}(s^*)}\right], \tag{16}$$

I being the nxn unit matrix.

Therefore, the displacements of the constrained (i.e., supported) beam can be written by using equation (9) as

$$w_{cons}(x,t) = \bar{w}_{cons}(x)e^{i\Omega t} \tag{17}$$

where

$$\bar{w}_{cons}(x) = \sum_{r=1}^{n} w_r(x)\bar{\eta}_r\,. \tag{18}$$

It is easy to show that the above expression can be reformulated as

$$\bar{w}_{cons}(x) = \left(\mathbf{w}^T(x)\,\mathbf{H}_{cons}(\Omega)\,\mathbf{w}(\gamma L)\right)F_0 \tag{19}$$

which in turn, after some rearrangements, leads to

$$\frac{\bar{w}_{cons}(x)}{\dfrac{F_0}{EI/L^3}} = \mathbf{a}^T(x)\,\mathbf{diag}\left(\frac{1}{\bar{\beta}_i^4 - \Omega^{*2}}\right)\mathbf{H}^*\left[\mathbf{I} - \frac{\mathbf{a}(s^*)\,\mathbf{a}^T(s^*)\,\mathbf{diag}\left(\dfrac{1}{\bar{\beta}_i^4 - \Omega^{*2}}\right)\mathbf{H}^*}{\mathbf{a}^T(s^*)\,\mathbf{diag}\left(\dfrac{1}{\bar{\beta}_i^4 - \Omega^{*2}}\right)\mathbf{H}^*\mathbf{a}(s^*)}\right]\mathbf{a}(\gamma L) \tag{20}$$

where

$$\Omega^* = \frac{\Omega}{\omega_0}, \qquad \omega_0^2 = \frac{EI}{mL^4},$$

$$\mathbf{w}^T(x) = \frac{1}{\sqrt{mL}}\mathbf{a}^T(x) = \frac{1}{\sqrt{mL}}\left[a_1(x)\ \ldots\ a_n(x)\right],$$

$$a_i(x) = \cosh\bar{\beta}_i\frac{x}{L} - \cos\bar{\beta}_i\frac{x}{L} - \bar{\eta}_i^*\left(\sinh\bar{\beta}_i\frac{x}{L} - \sin\bar{\beta}_i\frac{x}{L}\right),$$

$$\bar{\eta}_i^* = \frac{\cosh\bar{\beta}_i + \cos\bar{\beta}_i}{\sinh\bar{\beta}_i + \sin\bar{\beta}_i}, \qquad \bar{c} = \frac{c}{mL\omega_0},$$

$$\mathbf{H}^* = \mathbf{I} - \frac{i\bar{c}\,\Omega^*\mathbf{a}(\alpha L)\mathbf{a}^T(\alpha L)\,\mathbf{diag}\left(\dfrac{1}{\bar{\beta}_i^4 - \Omega^{*2}}\right)}{1 + i\bar{c}\,\Omega^*\mathbf{a}^T(\alpha L)\,\mathbf{diag}\left(\dfrac{1}{\bar{\beta}_i^4 - \Omega^{*2}}\right)\mathbf{a}(\alpha L)}. \tag{21}$$

Noting that according to equation (17), the real part of $\bar{w}_{cons}(x)\,e^{i\Omega t}$ represents the physical displacements, the amplitude distribution A(x) along the supported beam subject to the harmonic force is obtained as

$$A(x) = \sqrt{\bar{w}_{cons}^2(x)_{Re} + \bar{w}_{cons}^2(x)_{Im}} \tag{22}$$

In the case F_0 =1, the right hand side of equation (22) represents nothing else but the frequency response function of the beam in Figure 1.

Solution Through The Boundary Value Problem Formulation

In order to prove the validity of the expression (22) along with equations (20) and (21), the only way is to compare this with the results of a boundary value problem formulation.

The bending vibrations of the four beam portions shown in Figure 1 are governed by the partial differential equations

$$EIw_i^{IV}(x,t) + m\ddot{w}_i(x,t) = 0\,, \quad (i = 1,2,3,4) \tag{23}$$

with the following boundary and matching conditions:

$$w_1(0,t)=w_1'(0,t)=0\,,\quad w_1(s^*,t)=w_2(s^*,t)=0\,,$$

$$w_1'(s^*,t)=w_2'(s^*,t)\,,\quad w_1''(s^*,t)=w_2''(s^*,t)\,,$$

$$w_2(\alpha L,t)=w_3(\alpha L,t)\,,\quad w_2'(\alpha L,t)=w_3'(\alpha L,t)\,,$$

$$w_2''(\alpha L,t)=w_3''(\alpha L,t)\,,$$

$$EI\,w_2'''(\alpha L,t)\text{-}EI\,w_3'''(\alpha L,t)-c\dot{w}_2(\alpha L,t)=0\,,$$

$$w_3(\gamma L,t)=w_4(\gamma L,t)\,,\quad w_3'(\gamma L,t)=w_4'(\gamma L,t)\,,$$

$$w_3''(\gamma L,t)=w_4''(\gamma L,t)\,,\quad w_4''(L,t)=w_4'''(L,t)=0\,,$$

$$EI\,w_3'''(\gamma L,t)\text{-}EI\,w_4'''(\gamma L,t)+F_0e^{i\Omega t}=0\,,\qquad (24)$$

If harmonic solutions of the form

$$w_i(x,t)=W_i(x)\,e^{i\Omega t}\qquad (25)$$

are substituted into equation (23), the following ordinary differential equations are obtained for the amplitude functions $W_i(x)$:

$$W_i^{IV}(x)-\overline{\Lambda}^4W_i(x)=0\,,\quad (i = 1,2,3,4)\qquad (26)$$

where

$$\overline{\Lambda}^4=\frac{m\Omega^2}{EI}\qquad (27)$$

In the expressions above, both $w_i(x,t)$ and $W_i(x)$ represent complex valued functions. The essential point here is to imagine the actual bending displacements $w_i(x,t)$ as the real parts of some complex valued functions, for which the same notation is used for the sake of briefness.

The corresponding boundary and matching conditions read now

$$W_1(0)=W_1'(0)=0\,,\quad W_1(s^*)=W_2(s^*)=0\,,\quad W_1'(s^*)=W_2'(s^*)\,,$$

$$W_1''(s^*)=W_2''(s^*)\,,\quad W_2(\alpha L)=W_3(\alpha L)\,,\quad W_2'(\alpha L)=W_3'(\alpha L)\,,$$

$$W_2''(\alpha L)=W_3''(\alpha L)\,,\quad W_2'''(\alpha L)-W_3'''(\alpha L)-\frac{ic\,\Omega}{EI}W_2(\alpha L)=0\,,$$

$$W_3(\gamma L)=W_4(\gamma L)\,,\quad W_3'(\gamma L)=W_4'(\gamma L)\,,\quad W_3''(\gamma L)=W_4''(\gamma L)\,,$$

$$W_4''(L)=W_4'''(L)=0\,,\quad W_3'''(\gamma L)\text{-}W_4'''(\gamma L)+\frac{F_0}{EI}=0\,,\qquad (28)$$

The general solutions of the differential equations (26) are

$$W_1(x)=c_1\sin\overline{\Lambda}x+c_2\cos\overline{\Lambda}x+c_3\sinh\overline{\Lambda}x+c_4\cosh\overline{\Lambda}x\,,$$

$$W_2(x)=c_5\sin\overline{\Lambda}x+c_6\cos\overline{\Lambda}x+c_7\sinh\overline{\Lambda}x+c_8\cosh\overline{\Lambda}x\,,$$

$$W_3(x)=c_9\sin\overline{\Lambda}x+c_{10}\cos\overline{\Lambda}x+c_{11}\sinh\overline{\Lambda}x+c_{12}\cosh\overline{\Lambda}x\,,$$

$$W_4(x)=c_{13}\sin\overline{\Lambda}x+c_{14}\cos\overline{\Lambda}x+c_{15}\sinh\overline{\Lambda}x+c_{16}\cosh\overline{\Lambda}x\qquad (29)$$

where c_1 to c_{16} are unknown integration constants to be determined which can be complex in general.

Substitution of the expressions (29) into the conditions (28) yields, after rearrangement, the following set of sixteen inhomogeneous equations for the determination of the coefficients c_i :

$$\mathbf{A\,c=b.}\qquad (30)$$

The expression of the 16x16 coefficient matrix **A** is given in the Appendix. The vectors **c** and **b** are defined as

$$\mathbf{c}^T=\begin{bmatrix}c_1 & c_2 & \dots & c_{16}\end{bmatrix}$$

$$\mathbf{b}^T=\begin{bmatrix}0 & \dots & 0 & -\dfrac{F_0}{EI\,\overline{\Lambda}^3} & 0 & 0\end{bmatrix}\qquad (31)$$

where only the fourteenth element of the 16x1 vector **b** is non-zero.

Lengthy expressions of the elements c_i of the vector **c**, which were obtained by MATHEMATICA via symbolic computation, are not given here due to space limitations. However, it is important to note that the vector **c** and therefore the amplitude functions $W_i(x)$, (i = 1,2,3,4) in equations (29) contain the common factor $\dfrac{F_0}{EI/L^3}$ which has the dimension of length.

Having obtained $W_i(x)$ (i = 1,2,3,4), it is possible to determine the steady state amplitude at any point x of the beam, due to the harmonic force at a point $x = \gamma L$. Noting that according to equation (25) the real part of $W_i(x)\,e^{i\Omega t}$ represents the physical displacements, the amplitude distribution $\overline{\mathbf{A}}(x)$ along the supported beam subjected to the harmonically varying vertical force at $x = \gamma L$ is obtained as

$$\overline{A}(x)=\sqrt{W_i^2(x)_{Re}+W_i^2(x)_{Im}}\qquad (32)$$

In the case F_0=1, the right hand side of the above equation represents the frequency response function of the beam in Figure 1.

NUMERICAL APPLICATIONS

This section is devoted to the numerical evaluations of the formulae established in the preceding sections. In these examples, Ω*= 5 and $\bar{c} = 0.5$ are chosen. These mean that a harmonically varying vertical force of the radian frequency $5\sqrt{EI/mL^4}$ is acting at the location x = γL, shown in Figure 1, and the non-dimensionalized damping value is 0.5.

In the first example, the following data α = 0.75 and γ = 1.0 are chosen which mean that the damper and the harmonic force act at the points x = 0.75L and at the tip, respectively.

The displacement amplitudes at various sections of the beam, non-dimensionalized by dividing by $\frac{F_0}{EI/L^3}$ are given in Table 1. η represents the non-dimensional position of the support, whereas $\bar{x} = \frac{x}{L}$ denotes the non-dimensional position of the point, the vibration amplitude of which we are interested in. The values in the first columns are values obtained from formula (22), where n = 15 is taken in the series expansion (4) and $\bar{\beta}_1$ to $\bar{\beta}_{15}$ in equation (21) taken from reference [4] are correct up to twelve decimal places. These explanations are also valid for Tables 2 and 3. The values in the second columns are "exact" values from equation (32), obtained by the direct solution of the boundary value problem outlined in section 2.2.

		η					
		0,25		0,50		0,75	
	0,1	0.006068	0.006066	0.001378	0.001379	0.000587	0.000587
	0,2	0.008123	0.008086	0.004134	0.004132	0.001982	0.001981
	0,3	0.017457	0.017534	0.006193	0.006191	0.003636	0.003636
	0,4	0.079158	0.079275	0.005498	0.005495	0.005009	0.005008
$\bar{x}$	0,5	0.170389	0.170566	0	0	0.005564	0.005565
	0,6	0.284124	0.284393	0.011731	0.011736	0.004786	0.004785
	0,7	0.413805	0.414183	0.028777	0.028787	0.002163	0.002160
	0,8	0.553609	0.554087	0.049656	0.049666	0.002774	0.002774
	0,9	0.698720	0.699298	0.072943	0.072959	0.009737	0.009739
	1,0	0.845716	0.846386	0.097352	0.097378	0.017717	0.017725

Table 1. Dimensionless vibration amplitudes at various sections of the beam due to the harmonic forcing $F_0 e^{i\Omega t}$ at γ = 1.0. $\Omega = 5\sqrt{EI/mL^4}$ and α = 0.75 are chosen.

The second example is based on the data η = 0.25 and γ = 1.0 which in turn mean that the beam is supported at x = 0.25L and the harmonic force acts again at the tip. The non-dimensionalized vibration amplitudes at various sections of the beam are given in Table 2 for three different attachment points of the viscous damper to the beam : x = 0.25L, 0.50L and 0.75L. The values in the first and second columns are again values obtained from equations (22) and (32).

		α					
		0,25		0,50		0,75	
	0,1	0.008465	0.008470	0.008397	0.008401	0.006068	0.006066
	0,2	0.011333	0.011292	0.011242	0.011200	0.008123	0.008086
	0,3	0.024355	0.024484	0.024159	0.024286	0.017457	0.017534
	0,4	0.110440	0.110699	0.109550	0.109804	0.079158	0.079275
$\bar{x}$	0,5	0.237734	0.238186	0.235817	0.236258	0.170389	0.170562
	0,6	0.396433	0.397152	0.393239	0.393939	0.284124	0.284393
	0,7	0.577387	0.578415	0.572738	0.573739	0.413805	0.414183
	0,8	0.772457	0.773794	0.766244	0.767544	0.553609	0.554087
	0,9	0.974908	0.976559	0.967073	0.968679	0.698720	0.699298
	1,0	1.179962	1.181920	1.170488	1.172391	0.845716	0.846386

Table 2. Dimensionless vibration amplitudes at various sections of the beam due to the harmonic forcing $F_0 e^{i\Omega t}$ at γ = 1.0. $\Omega = 5\sqrt{EI/mL^4}$ and η = 0.25 are chosen.

And finally, the third example is concerned with η = 0.25 and α = 0.50, i.e., the beam is supported at x = 0.25L and the damper attachment point is the midpoint of the beam. The non-dimensionalized amplitudes at various beam sections are given in Table 3 for three acting points of the harmonic force on to the beam : x = 0.50L, 0.75L and L. The first columns are values obtained from equation (22) whereas those of the second columns are determined by equation (32).

		γ					
		0,50		0,75		1,00	
	0,1	0.001863	0.001864	0.004933	0.004936	0.008397	0.008401
	0,2	0.002495	0.002485	0.006605	0.006580	0.011242	0.011200
	0,3	0.005346	0.005376	0.014186	0.014262	0.024159	0.024286
	0,4	0.023953	0.024013	0.064187	0.064338	0.109550	0.109804
$\bar{x}$	0,5	0.050687	0.050791	0.137736	0.137999	0.235817	0.236258
	0,6	0.082927	0.082925	0.228812	0.229229	0.393239	0.393939
	0,7	0.118851	0.119083	0.331769	0.332364	0.572738	0.573739
	0,8	0.157010	0.157313	0.441561	0.442337	0.766244	0.767544
	0,9	0.196253	0.196626	0.554433	0.555390	0.967073	0.968679
	1,0	0.235817	0.236258	0.668217	0.669348	1.170488	1.172391

Table 3. Dimensionless vibration amplitudes at various sections of the beam due to the harmonic forcing $F_0 e^{i\Omega t}$ at three acting points. $\Omega = 5\sqrt{EI/mL^4}$, η = 0.25 and α = 0.50 are chosen.

The agreement of the values in both columns in Tables 1 to 3 justifies the expression (22) along with equations (20) and (21), obtained on the basis of a formula established for the receptance matrix of viscously damped discrete systems subject to several constraint equations. It is worth nothing that the agreements of the numbers in both columns become excellent if many more decimal places are considered in $\bar{\beta}_i$ values.

CONCLUSIONS

This study is concerned with the determination of the frequency response function of a viscously damped, cantilevered Bernoulli-Euler beam, which is simply supported in-span. The frequency response function is obtained through a formula, which was established for the receptance matrix of discrete systems subjected to linear constraint equations. The comparison of the numerical results obtained with those via a boundary value problem formulation justifies the approach used here.

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APPENDIX

The matrix **A** in equation (30) :

$$
\mathbf{A}=\begin{vmatrix}
0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\mathrm{Sin}\,\bar{\Lambda}\,\eta L & \mathrm{Cos}\,\bar{\Lambda}\,\eta L & \sinh\bar{\Lambda}\,\eta L & \cosh\bar{\Lambda}\,\eta L & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \sin\bar{\Lambda}\,\eta L & \cos\bar{\Lambda}\,\eta L & \sinh\bar{\Lambda}\,\eta L & \cosh\bar{\Lambda}\,\eta L & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\cos\bar{\Lambda}\,\eta L & -\sin\bar{\Lambda}\,\eta L & \cosh\bar{\Lambda}\,\eta L & \sinh\bar{\Lambda}\,\eta L & -\cos\bar{\Lambda}\,\eta L & \sin\bar{\Lambda}\,\eta L & -\cosh\bar{\Lambda}\,\eta L & -\sinh\bar{\Lambda}\,\eta L & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-\sin\bar{\Lambda}\,\eta L & -\cos\bar{\Lambda}\,\eta L & \sinh\bar{\Lambda}\,\eta L & \cosh\bar{\Lambda}\,\eta L & \sin\bar{\Lambda}\,\eta L & \cos\bar{\Lambda}\,\eta L & -\sinh\bar{\Lambda}\,\eta L & -\cosh\bar{\Lambda}\,\eta L & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \sin\bar{\Lambda}\,\alpha L & \cos\bar{\Lambda}\,\alpha L & \sinh\bar{\Lambda}\,\alpha L & \cosh\bar{\Lambda}\,\alpha L & -\sin\bar{\Lambda}\,\alpha L & -\cos\bar{\Lambda}\,\alpha L & -\sinh\bar{\Lambda}\,\alpha L & -\cosh\bar{\Lambda}\,\alpha L & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \cos\bar{\Lambda}\,\alpha L & -\sin\bar{\Lambda}\,\alpha L & \cosh\bar{\Lambda}\,\alpha L & \sinh\bar{\Lambda}\,\alpha L & -\cos\bar{\Lambda}\,\alpha L & \sin\bar{\Lambda}\,\alpha L & -\cosh\bar{\Lambda}\,\alpha L & -\sinh\bar{\Lambda}\,\alpha L & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -\sin\bar{\Lambda}\,\alpha L & -\cos\bar{\Lambda}\,\alpha L & \sinh\bar{\Lambda}\,\alpha L & \cosh\bar{\Lambda}\,\alpha L & \sin\bar{\Lambda}\,\alpha L & \cos\bar{\Lambda}\,\alpha L & -\sinh\bar{\Lambda}\,\alpha L & -\cosh\bar{\Lambda}\,\alpha L & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & A_1 & A_2 & A_3 & A_4 & \cos\bar{\Lambda}\,\alpha L & -\sin\bar{\Lambda}\,\alpha L & -\cosh\bar{\Lambda}\,\alpha L & -\sinh\bar{\Lambda}\,\alpha L & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \sin\bar{\Lambda}\,\gamma L & \cos\bar{\Lambda}\,\gamma L & \sinh\bar{\Lambda}\,\gamma L & \cosh\bar{\Lambda}\,\gamma L & -\sin\bar{\Lambda}\,\gamma L & -\cos\bar{\Lambda}\,\gamma L & -\sinh\bar{\Lambda}\,\gamma L & -\cosh\bar{\Lambda}\,\gamma L \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cos\bar{\Lambda}\,\gamma L & -\sin\bar{\Lambda}\,\gamma L & \cosh\bar{\Lambda}\,\gamma L & \sinh\bar{\Lambda}\,\gamma L & -\cos\bar{\Lambda}\,\gamma L & \sin\bar{\Lambda}\,\gamma L & -\cosh\bar{\Lambda}\,\gamma L & -\sinh\bar{\Lambda}\,\gamma L \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\sin\bar{\Lambda}\,\gamma L & -\cos\bar{\Lambda}\,\gamma L & \sinh\bar{\Lambda}\,\gamma L & \cosh\bar{\Lambda}\,\gamma L & \sin\bar{\Lambda}\,\gamma L & \cos\bar{\Lambda}\,\gamma L & -\sinh\bar{\Lambda}\,\gamma L & -\cosh\bar{\Lambda}\,\gamma L \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\cos\bar{\Lambda}\,\gamma L & \sin\bar{\Lambda}\,\gamma L & \cosh\bar{\Lambda}\,\gamma L & \sinh\bar{\Lambda}\,\gamma L & \cos\bar{\Lambda}\,\gamma L & -\sin\bar{\Lambda}\,\gamma L & -\cosh\bar{\Lambda}\,\gamma L & -\sinh\bar{\Lambda}\,\gamma L \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\sin\bar{\Lambda}\,L & -\cos\bar{\Lambda}\,L & \sinh\bar{\Lambda}\,L & \cosh\bar{\Lambda}\,L \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\cos\bar{\Lambda}\,L & \sin\bar{\Lambda}\,L & \cosh\bar{\Lambda}\,L & \sinh\bar{\Lambda}\,L
\end{vmatrix}
$$

Where

$$A_1=-i\frac{c\,\Omega}{EI\bar{\Lambda}^3}\sin\bar{\Lambda}\,\alpha L-\cos\bar{\Lambda}\,\alpha L\ ,\qquad A_2=-i\frac{c\,\Omega}{EI\bar{\Lambda}^3}\cos\bar{\Lambda}\,\alpha L+\sin\bar{\Lambda}\,\alpha L\ ,$$

$$A_3=-i\frac{c\,\Omega}{EI\bar{\Lambda}^3}\sinh\bar{\Lambda}\,\alpha L+\cosh\bar{\Lambda}\,\alpha L\ ,\qquad A_4=-i\frac{c\,\Omega}{EI\bar{\Lambda}^3}\cosh\bar{\Lambda}\,\alpha L+\sinh\bar{\Lambda}\,\alpha L\ .$$