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ON THE NODES OF A CRACKED CANTILEVER BEAM

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ABSTRACT

Many researchers reported the use of vibration methods to evaluate crack in beam structures by comparing vibration amplitude data between the cracked and the uncracked states of the structures. But, these methods are not applicable if modal data is not available in prior or the crack is around a nodal point. In this study, a vibration method, which requires only the data of nodes for the cracked state of the beam, are proposed. In this paper, transverse vibrations of a cantilever beam with a transverse crack are investigated analytically and additionally, the Bernoulli-Euler beam theory is employed to characterise transverse surface crack in beams. In formulations, the beam has a uniform cross-section and the crack was modelled by sawing cuts. In order to model the transverse vibration, the crack is represented by a rotational spring. It is well known that when a crack develops in a component, it leads to changes in its mode shapes. The cantilever beam is subjected to a transverse harmonic force in the vicinity of the node. The changes of the nodes for the second and third mode are obtained as a function of the crack location and size. The steady-state amplitude of motion of the excitation point is computed, and its variation as a function of the forcing frequency, crack location and size is investigated. Frequency spectra for the response of the uncracked and cracked beam are compared to each other under harmonic forcing.

INTRODUCTION

It is well known that when a crack develops in a component it leads to changes in its vibration parameters. Hence it may be possible to estimate the location and size of the crack by measuring the changes in vibration parameters. Several methods have been developed for non-destructive testing using vibration techniques. Measurements of amplitudes, natural frequencies and vibration modes have been used in the identification of location and the magnitude of a crack in a beam.

Cracked structures have been modelled by various methods. Dimarogonas and Papadopoulos have computed the flexibility matrix for a transverse surface crack on a shaft in reference [1]. Dentsoros and Dimarogonas in reference [2] analysed a cantilevered beam with a crack at the fixed end. A longitudinal harmonic force was applied at the free end, and the crack was modelled as a linear spring. Chondros and Dimarogonas in reference [3], Dimarogonas and Massouros in reference [4], combined this spring hinge model with fracture mechanics results, and developed a frequency spectral method to identify cracks in various structures. For a known crack position, this method correlated the crack depth to the change in natural frequencies of the first three harmonics of the structure. Dimarogonas and Papadopoulos considered coupled longitudinal and bending vibrations of a cantilevered beam with a transverse crack. Using a spring model for the crack, the first three modes of free vibration were determined. In the analysis, the beam was given a longitudinal, harmonic displacement at its fixed end and the response was plotted as a function of the excitation frequency in reference [5]. Rizos, Aspragathos and Dimarogonas in reference [6] used the analytical results to relate the measured flexural amplitudes at two points of the structure vibrating at one of its natural modes to the crack location and depth. Gounaris and Dimarogonas in reference [7] used FEM to evaluate the dynamic response of a cracked cantilever beam to harmonic point force excitation. Mermertaş and Erol in reference [8] investigated the effect of mass attachment on the free vibration of cracked beam. Collins, Plaut and Wauer in reference [9] analysed free and forced longitudinal vibrations of a cantilever bar with a crack. The steady-state amplitude of the motion of the free end was plotted as a function of the forcing frequency. Thus, the location and the compliance of the crack, and frequency spectra were obtained. Bamnios and Trochides in reference [10] investigated the influence of a transverse surface crack on the dynamic behaviour of a cantilever beam both analytically and

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experimentally. Modelling the crack as a rotational spring, relations linking the change in natural frequencies and in mechanical impedance to the location and depth of the crack were obtained. An extended literature review of these methods can be seen in reference [11]. Ishak, Liu and Lim in reference [12] conducted strip element method calculations and experimental results to identify the crack location. In the theoretical analysis, the beam is divided into domains and a harmonic load is applied on its surface.

In this study, modelling the crack as a rotational spring, relations linking the change in nodes and frequency spectra to the location and depth of the crack are obtained. It was assumed that the crack remained open during vibration. The cantilever beam is subjected to a transverse harmonic force in the vicinity of the node. The changes of the nodes for the second and third mode are obtained as a function of the crack location and size. The steady-state amplitude of the motion of the excitation point is computed, and its variation as the function of the forcing frequency, the crack location and size are investigated. Frequency spectra for the response of the uncracked and cracked beam under harmonic forcing are compared to each other. The method is based on the examination of the change in nodes can be used for crack identification in simple beam structures.

THEORY

In order to model the transverse vibrations, the crack is represented by a rotational spring of stiffness K_T. The existence of a crack in a beam increases the local flexibility of the beam. Dimaragonas and Paipetis in reference [13] calculated the bending spring constant K_T in the vicinity of the cracked section of a beam with orthogonal cross-section of width b and the height h as can be seen in Figure 1, when a lateral crack of uniform depth a exist, from the crack strain energy function;

$$K_T = \frac{E\,I}{6\pi(1-\vartheta^2)\,h\,\Phi(a/h)} \tag{1}$$

where E is the modulus of elasticity of a beam material, I is the moment of inertia of the beam cross-section. The dimensionless local compliance function $\Phi(a/h)$ is computed from the strain energy density function and has the form,

$$\begin{aligned}\Phi(a/h) = {} & 0.6272\,(a/h)^2 - 1.04533\,(a/h)^3 + 4.5948\,(a/h)^4 \\ & - 9.973\,(a/h)^5 + 20.2948\,(a/h)^6 - 33.0351\,(a/h)^7 \\ & + 47.1063\,(a/h)^8 - 40.755\,(a/h)^9 + 19.6\,(a/h)^{10}\end{aligned} \tag{2}$$

The bending vibrations of a uniform Bernoulli-Euler beam are governed by the partial differential equation**,**

$$E\,I\frac{\partial^4 w(x,t)}{\partial x^4} + \rho A \frac{\partial^2 w(x,t)}{\partial t^2} = 0 \tag{3}$$

where w(x,t) denotes the lateral displacement at point x at time t. A is the cross-sectional area and ρ is the mass density.

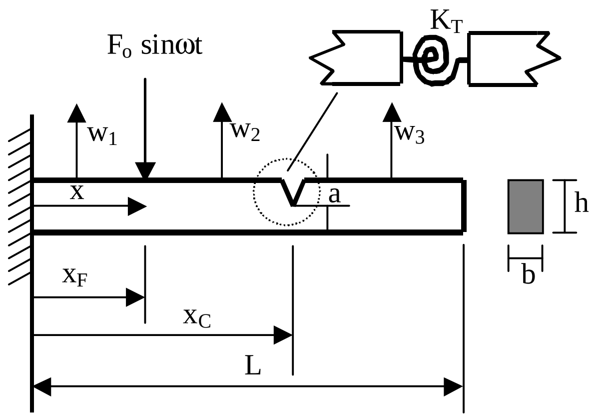

Figure 1. Cracked cantilever beam subjecting a harmonic force

The locations of the driving force and of the crack divide the beam in three parts. Depending on the location of the driving force one has to solve two problems with different boundary conditions. If the force is between crack and clamped end, the bending displacements in the regions to the left and right of the in-span excitation will be denoted as $w_1(x,t)$ and $w_2(x,t)$ where both are subject to the differential equation (3). The corresponding matching conditions at the excitation,

$$\begin{gathered} w_1(\beta_F,t) = w_2(\beta_F,t)\,, \quad w_1'(\beta_F,t) = w_2'(\beta_F,t)\,, \\ w_1''(\beta_F,t) = w_2''(\beta_F,t)\,, \quad E\,I\left(w_1'''(\beta_F,t) - w_2'''(\beta_F,t)\right) = F_o\ \sin\omega t \end{gathered} \tag{4}$$

where

$$\beta_F = \frac{x_F}{L}$$

The crack is assumed to be open and to have uniform depth. The beam can be conveniently divided into two segments, one on either side of the spring representing the crack. The bending displacements in the regions to the left and right of the crack will be denoted as $w_2(x,t)$ and $w_3(x,t)$. The continuity of displacement, moment and shear forces at the crack location ($\beta_C = \frac{x_C}{L}$) and jump condition in the slope can be written in the following form,

$$\begin{gathered} w_2(\beta_C,t) = w_3(\beta_C,t)\,, \quad w_2''(\beta_C,t) = w_3''(\beta_C,t) \\ w_2'''(\beta_C,t) = w_3'''(\beta_C,t)\,, \\ w_3'(\beta_C,t) = w_2'(\beta_C,t) + E\,I\,w_3''(\beta_C,t)/K_T \end{gathered} \tag{5}$$

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If the driving force is between free end and crack, the boundary conditions are,

$$
\begin{gathered}
w_1(\beta_C,t)=w_2(\beta_C,t), \quad w_1''(\beta_C,t)=w_2''(\beta_C,t),\\
w_1'''(\beta_C,t)=w_2'''(\beta_C,t),\\
w_2'(\beta_C,t)=w_1'(\beta_C,t)+E\,I\,w_2''(\beta_C,t)/K_T\ ,\\
w_2(\beta_F,t)=w_3(\beta_F,t), \quad w_2'(\beta_F,t)=w_3'(\beta_F,t),\\
w_2''(\beta_F,t)=w_3''(\beta_F,t),\\
E\,I\left(w_2'''(\beta_F,t)-w_3'''(\beta_F,t)\right)=F_o \sin\omega t
\end{gathered}
\tag{6}
$$

The boundary conditions at the fixed and free ends, respectively, are

$$
\begin{gathered}
w_1(0,t)=0\,, \quad w_1'(0,t)=0\,,\\
w_3''(1,t)=0\,, \quad w_3'''(1,t)=0\,.
\end{gathered}
\tag{7}
$$

If harmonic solutions are assumed for three regions,

$$
w_i(x,t)=Y_i(x)\sin\omega t\,, \quad i = 1,2,3. \tag{8}
$$

The displacement on each part of the beam is assumed as

$$
\begin{aligned}
Y_1(\beta)&=A_1\cos(\lambda\beta)+A_2\sin(\lambda\beta)+A_3\cosh(\lambda\beta)+A_4\sinh(\lambda\beta)\\
Y_2(\beta)&=A_5\cos(\lambda\beta)+A_6\sin(\lambda\beta)+A_7\cosh(\lambda\beta)+A_8\sinh(\lambda\beta)\\
Y_3(\beta)&=A_9\cos(\lambda\beta)+A_{10}\sin(\lambda\beta)+A_{11}\cosh(\lambda\beta)+A_{12}\sinh(\lambda\beta)
\end{aligned}
\tag{9}
$$

where A_i i = 1,…,12, are arbitrary integration constant to be evaluated from the boundary and matching conditions for w_1, w_2 and w_3. The system given by equation (9) and boundary and matching conditions was solved and analytical expressions for the receptance at different driving points were obtained. In order to obtain non-vanishing solutions for A_1 to A_{12}, the corresponding determinant of coefficients has to be equated to zero yields the characteristic equation. This equality is satisfied for an infinite number of the system's natural frequencies. Substituting any one of these back into equation (9) yields a corresponding set of relative values for A_i i = 1,…,12, i.e., $\{A_i\}_r$, the so-called mode shape corresponding to that natural frequency. The following non-dimensional parameter is introduced,

$$
\lambda^4=\frac{\omega^2\,\rho\,A\,L^4}{E\,I}, \quad \beta=\frac{x}{L} \tag{10}
$$

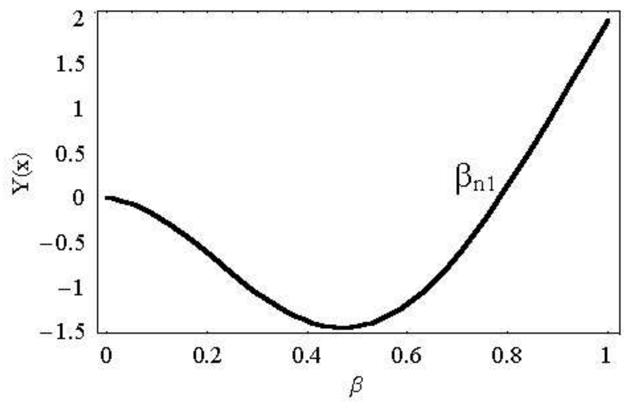
where, ω is the angular excitation frequency, x is the coordinate along the beam with the origin at the clamp end and L is the length of the beam.

NUMERICAL APPLICATION

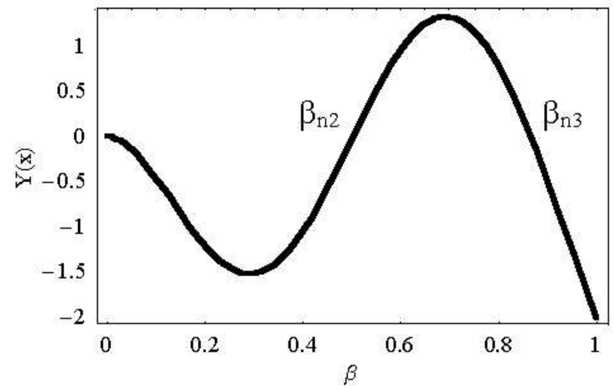
In Figure 2 second and third mode shapes and their nodes are viewed. The variations in the first three nodes versus the crack location parameter for a cracked cantilever beam with depth ratio a/h = 0.6 are given in Table 1. The values of the cracked beam obtained at the free end (β_C = 1) are the same with those of the uncracked beam.

β_C	0	0.250	0.500	0.775	1.000
β_{n1}	0.7733	0.7836	0.7825	0.7896	0.7834
β_{n2}	0.4838	0.4966	0.5032	0.5153	0.5035
β_{n3}	0.8624	0.8651	0.8676	0.8715	0.8676

Table 1. The variation in the first three nodes with respect to the crack location parameter for a cantilever cracked beam with depth ratio a/h = 0.6.



(a) The second mode.



(b) The third mode.
Figure 2. Plot of the second and third mode shapes of the cantilever beam.

For forced vibration, the variations in receptance on a logarithmic scale for second and third mode are depicted as a

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function of non-dimensional forcing frequency in Figure 3-8. The location of the node is chosen as the point for forcing. The crack location parameter of $\beta_C = 0.775$ together with a constant crack depth ratio a/h = 0.6 is considered. Calculation in this study was carried out via the following beam data: length 1m, height 0.01 m, width 0.01 m, Young's modulus $E=2.11x10^{11}$Pa, Poisson ratio $\vartheta = 0.3$, density $\rho = 7860$ kg/m^3. If the beam is uncracked, when it is excited at its first node, a peak with the receptance value is not seen at the point where we accepted to see the second natural frequency. If the beam is cracked, a jump is observed in the receptance with a lower frequency (Figure 3.) This result is also valid for the other two nodes of the third mode shape (Figures 5 and 7.)

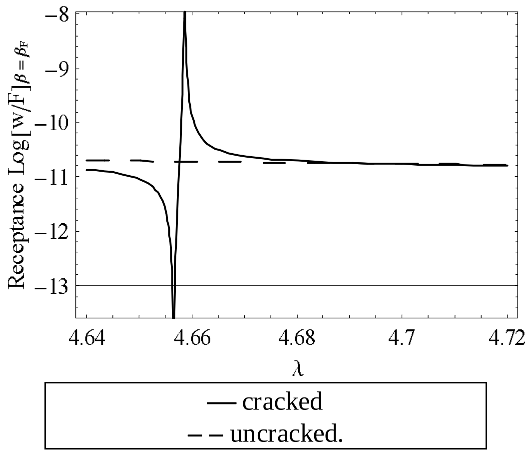

Figure 3. Receptance plots for cracked and uncracked cantilever beams with depth ratio a/h = 0.6
($\beta_C = 0.775$, $\beta_F = \beta_{n1(uncracked)} = 0.7834$).

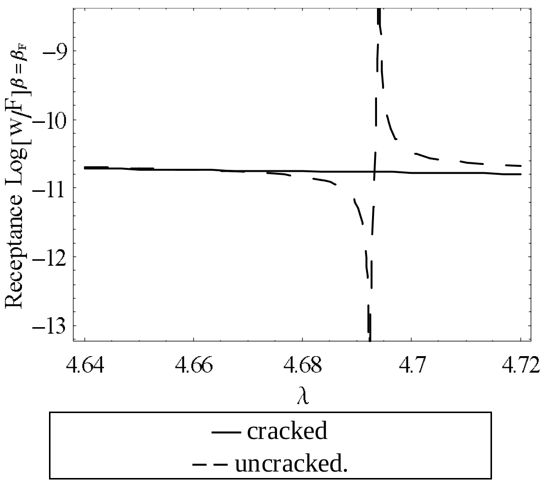

Figure 4. Receptance plots for cracked and uncracked cantilever beams with depth ratio a/h = 0.6
($\beta_C = 0.775$, $\beta_F = \beta_{n1(cracked)} = 0.7896$).

After this first step, where whether or not the beam is cracked is resolved, the location of the node on the cracked beam is determined. By moving the forcing point to the left and right, the point where peak disappears in the frequency response function is found. This process is repeated for all of the considered nodes at the observed frequency intervals. The receptance obtained at the new forcing points of the cracked beam depending on the frequency are drawn in Figures 4,6 and 8 in comparison with the uncracked beam. In this case, the second and the third natural frequencies of the uncracked beam are attained. Since the first mode shape does not include a node, there is no excitation point on the beam to remove the first natural frequency, whether the beam is cracked or not.

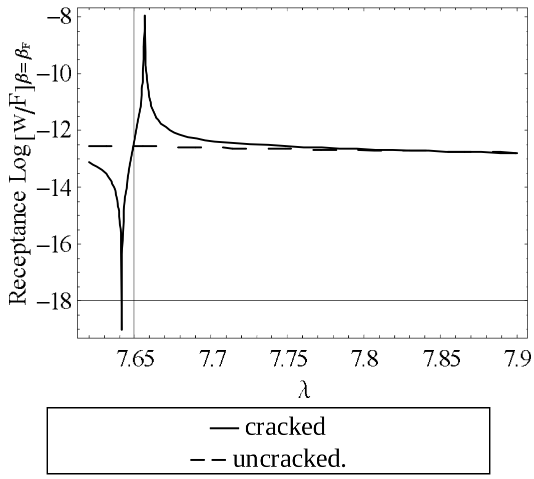

Figure 5. Receptance plots for cracked and uncracked cantilever beams with depth ratio a/h = 0.6
($\beta_C = 0.775$, $\beta_F = \beta_{n2(uncracked)} = 0.5035$).

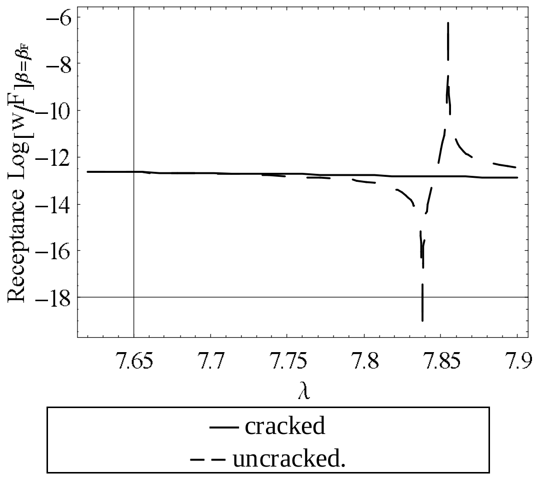

Figure 6. Receptance plots for cracked and uncracked cantilever beams with depth ratio a/h = 0.6
($\beta_C = 0.775$, $\beta_F = \beta_{n2(cracked)} = 0.5153$).

In Figure 9, the variation in the node of the second mode versus the location of the crack for the cantilever beams which have crack depth ratio a/h=0.4 and a/h=0.6 is given. In Figures

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10 and 11, changes in the two nodes of the third mode relative to the crack location on the cantilever beam is presented for the same cantilever beams compared to the uncracked beam. As the crack depth rises, the variation in the nodes is higher in the cracked beam than the uncracked beam. Although certain nodes do not change at some of the positions of the crack, there is no crack location where all of the nodes remain unchanged. Therefore, more than one node should be observed in order to determine whether the beam is cracked or uncracked.

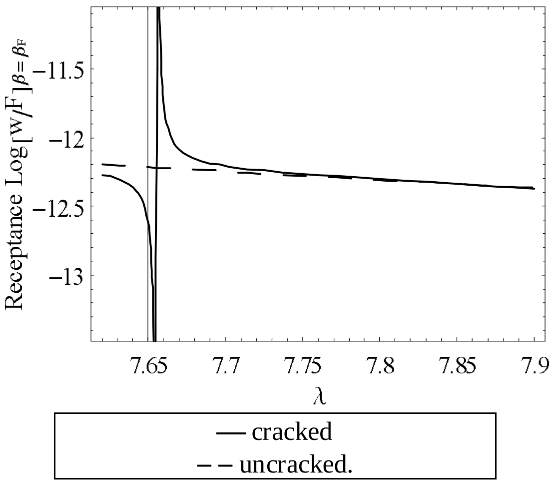

Figure 7. Receptance plots for cracked and uncracked cantilever beams with depth ratio a/h = 0.6 (β_C = 0.775, $\beta_F = \beta_{n3(uncracked)}$ = 0.8676).

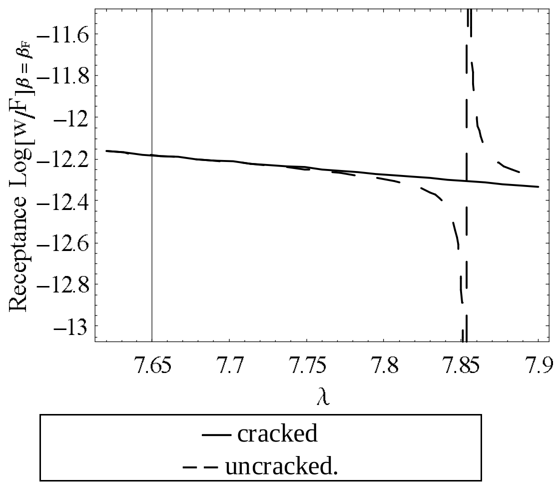

Figure 8. Receptance plots for cracked and uncracked cantilever beams with depth ratio a/h = 0.6 (β_C = 0.775, $\beta_F = \beta_{n3(craked)}$ = 0.8715).

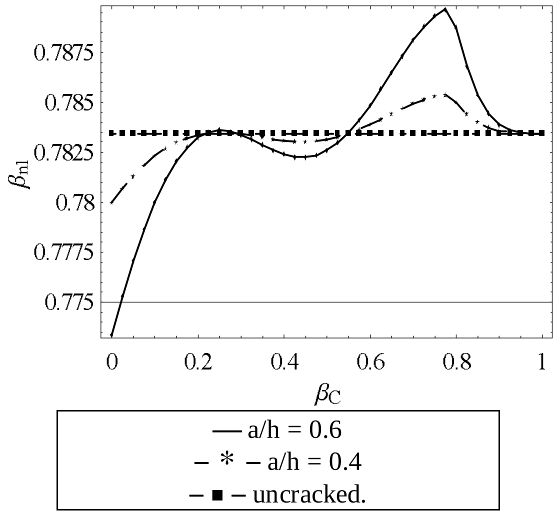

Figure 9. The variation in the first node (β_{n1}) versus the crack location parameter for a cracked cantilever beam.

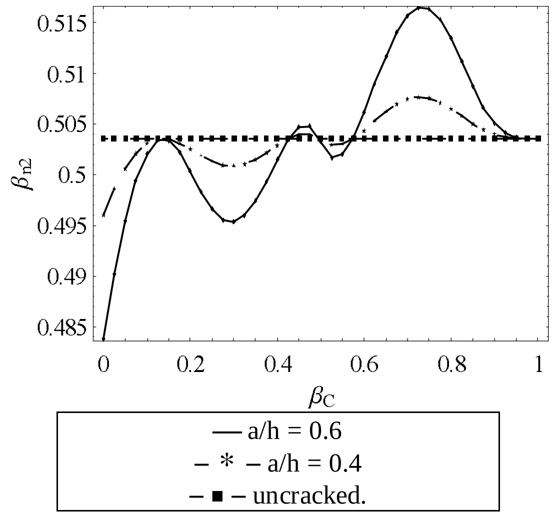

Figure 10. The variation in the second node (β_{n2}) versus the crack location parameter for a cracked cantilever beam.

The diagrams given in the Figures 9-11 are employed to determine the crack location and depth by solving the problem reverse. For this purpose, previously obtained three nodes are evaluated for the selected crack depth and the possible crack locations that give these nodes are determined. Given node can be met by more than one location. The correct result is the position that provides all of the three nodes. If there is no harmony between the probable points for the crack depth under scrutiny, then the same investigation is carried out for another crack depth. Crack depth and the size that gives the common point give the expected solution. For a more sensitive solution, similar comparisons for nodes at higher modes can be carried out with elevated frequencies.

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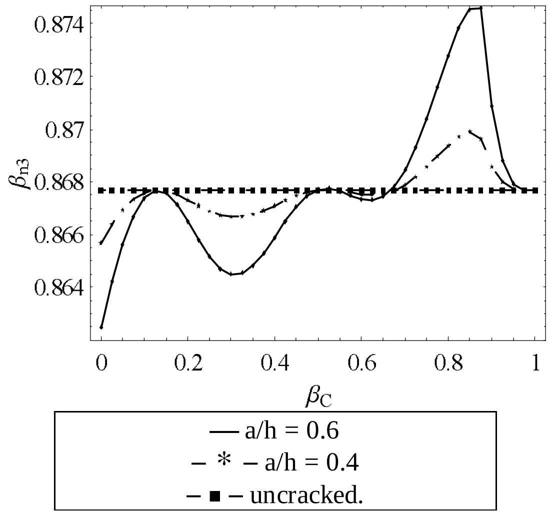

Figure 11. The variation in the third node (β_{n3}) versus the crack location parameter for a cracked cantilever beam.

CONCLUSIONS

The cantilever beam is subjected to a transverse harmonic force in the vicinity of the node. The changes of the nodes for the second and third mode are obtained as a function of the crack location and size. The steady-state amplitude of motion of the excitation point is computed, and its variation as a function of the forcing frequency, crack location and size is investigated. Frequency spectra for the response of the uncracked and cracked beam under harmonic forcing are compared to each other. The method is based on the examination of the change in nodes can be used for crack identification in simple beam structures. It would be useful to verify the results experimentally. The method is composed of the following steps:

1. If the beam is cracked, when excited from the first node of the uncracked beam, a peak will be observed signalling the second natural frequency. However, for the uncracked beam, a peak in the variation of the receptance depending on the frequency is not seen.

2. After this first step, where whether or not the beam is cracked is determined, the force application point is moved to the right and left in order to find out the point where the peak disappears in the frequency response function.

3. Three nodes of the second and the third mode shapes are evaluated for the selected crack depth. Given node can be met by more than one location. The crack location providing all of the three nodes is the searched solution.

4. If there is no harmony between the possible crack locations for the investigated crack depth, the same search is carried out for another crack depth.

5. For a more sensitive solution, similar comparisons for nodes at higher modes can be carried out with elevated frequencies.

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