LATTICE-LADDER STRUCTURE FOR 2D ARMA FILTERS

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ABSTRACT

A novel lattice-ladder structure for the realization of 2D ARMA digital filters is presented. The new realization is based on the 2D AR lattice filter. The algorithm to calculate the lattice-ladder structure coefficients for a given 2D ARMA transfer function is included. The 2D lattice-ladder structure has the properties of orthogonality and modularity as in the 1D case. The lattice-ladder structure might prove useful in 2D adaptive filtering applications.

1. INTRODUCTION

ARMA or pole-zero digital filters are important in that they can provide parsimonious yet efficient system models. 1D ARMA lattice-ladder structures have found applications in adaptive filtering and speech processing [1], [2]. The 1D ARMA lattice-ladder structure consists of an all-pole lattice section realizing the AR part of the system and the all-zero ladder section providing the MA part . The ladder section employs linear regression on the backward prediction errors generated by the lattice section. However, in the literature there is yet no compatible lattice-ladder structure for 2D ARMA digital filters, which are utilized for 2D digital filtering and digital image processing.

In this brief we develop a new lattice-ladder structure for the realization of 2D ARMA digital filters. This structure utilizes the 2D AR lattice model recently proposed in [3] as the backbone and juxtaposes a ladder section to this 2D AR model to create the full ARMA structure. In [4] a two-channel AR lattice approach for 2D ARMA lattice modelling was reported. However, this approach suffers from a four-fold increase in the number of reflection coefficients due to the multichannel structure. The model in this paper eliminates any redundancy from the lattice reflection coefficients. A recursive algorithm to calculate the lattice-ladder coefficients for any given 2D ARMA transfer function is also presented. In 1D filtering, lattice and lattice-ladder structures have been studied because of their advantages such as modularity, built-in stability and robustness to finite-word-length effects. The 2D lattice-ladder structure maintains the orthogonality of prediction errors and modularity properties of its 1D counterpart. Hence, this structure will be useful for adaptive filtering applications.

2. 2D LATTICE-LADDER MODEL

The system function for the 2D ARMA pole-zero model is given as follows:

$$\begin{aligned} H(z_1, z_2) =& \frac{Y(z_1, z_2)}{X(z_1, z_2)} = \frac{B(z_1, z_2)}{A(z_1, z_2)} \\ =& \frac{\sum\sum_{(n_1,n_2)\in\mathscr{R}} b(n_1, n_2) z_1^{-n_1} z_2^{-n_2}}{1 + \sum\sum_{(n_1,n_2)\in\mathscr{R}-(0,0)} a(n_1, n_2) z_1^{-n_1} z_2^{-n_2}} \end{aligned} \quad (1)$$

Here, $\mathscr{R}$ denotes the 2D region of support for the numerator and denominator polynomial parameters. Without loss of generality, we assume that the support for both polynomials is the same. In [3], a 2D orthogonal lattice structure for 2D AR models has been presented. In the synthesis mode this structure can be utilized to form a 2D AR random field with a given AR transfer function. This model simultaneously creates the orthogonal backward prediction errors corresponding to the 2D AR system model. A Levinson-type recursion to compute the 2D lattice filter reflection coefficients for a given 2D AR transfer function was also developed in [3]. We present a novel structure for 2D ARMA filters by adding a ladder section to the 2D AR model of [3]. The complete 2D lattice-ladder structure for a quarter-plane shaped support $\mathscr{R}$ of size $N_1 \times N_2$ is presented in Fig. 1. In this figure $M = N_1 \cdot N_2 - 1$. The scheme used for ordering the samples on the support region is also included in this figure. The internal structures of the basic lattice modules utilized in the lattice-ladder model are depicted in Fig. 2 and Fig. 3 for completeness. In Fig. 1, the lattice section realizes the AR part of the transfer function $\left(1/A(z_1, z_2)\right)$, whereas the ladder section realizes the MA part $\left(B(z_1, z_2)\right)$. Similar to the 1D case, the output of the overall ARMA system is formed by taking a weighted linear combination of the backward prediction errors, $b_p^{(p)}(n_1, n_2)$, where the weights are the ladder coefficients c_p, for $p = 0, 1 \ldots, M$:

$$y(n_1, n_2) = \sum_{p=0}^{M} c_p\, b_p^{(p)}(n_1, n_2) \tag{2}$$

3. CALCULATION OF COEFFICIENTS

We derive the algorithm to calculate the lattice and ladder coefficients necessary for the lattice-ladder realization of a given ARMA transfer function,

$$H(z_1, z_2) = \frac{B(z_1, z_2)}{A(z_1, z_2)} \tag{3}$$

In [3], a Levinson-type recursion to compute the reflection coefficients $\Gamma_{f_{p-n}}^{(n)}$ and $\Gamma_{b_p}^{(n)}$ by solving the 2D augmented normal equations is outlined. These lattice reflection coefficients realize the given AR transfer function.

$$H_{\text{AR}}(z_1, z_2) = \frac{1}{A(z_1, z_2)} = \frac{B_0^{(0)}(z_1, z_2)}{X(z_1, z_2)} \tag{4}$$

We start from this point and assume that the reflection coefficients for the lattice part are already determined using the results in [3]. It is now necessary to calculate the ladder coefficients c_p in Fig. 1, which will realize the MA part of the transfer function,

$$H_{\text{MA}}(z_1, z_2) = B(z_1, z_2) = \frac{Y(z_1, z_2)}{B_0^{(0)}(z_1, z_2)} \tag{5}$$

We need some definitions to this end. The backward prediction error transfer function $\left(G_p^{(p)}(z_1, z_2)\right)$ is defined as the transfer function between the input of the MA section $\left(\text{i.e. } b_0^{(0)}(n_1, n_2)\right)$, and the backward prediction error $\left(b_p^{(p)}(n_1, n_2)\right)$:

$$\begin{aligned} G_p^{(p)}(z_1, z_2) &= \frac{B_p^{(p)}(z_1, z_2)}{B_0^{(0)}(z_1, z_2)} \\ &= \sum\sum_{(n_1,n_2)\in\mathscr{R}} g_p^{(p)}(n_1, n_2)\, z_1^{-n_1} z_2^{-n_2} \end{aligned} \tag{6}$$

These backward prediction error transfer functions can be calculated using the step-up recursion formula in [3] and the lattice reflection coefficients. The coefficients for the backward prediction error transfer functions in (6) are defined as $g_p^{(p)}(n_1, n_2)$, $(n_1, n_2) \in \mathscr{R}$. We will also define the following transfer functions $D_m(z_1, z_2)$, for $m = 0, 1, \ldots, M$. These transfer functions will be important in devising the recursive algorithm for calculating the ladder coefficients.

$$\begin{aligned} D_m(z_1, z_2) &= \sum_{p=0}^{m} c_p\, G_p^{(p)}(z_1, z_2) \\ &= \sum\sum_{(n_1,n_2)\in\mathscr{R}} d_m(n_1, n_2)\, z_1^{-n_1} z_2^{-n_2} \end{aligned} \tag{7}$$

It is trivial to see from this definition that $D_m(z_1, z_2)$ can be computed recursively from the backward prediction error transfer functions.

$$D_m(z_1, z_2) = D_{m-1}(z_1, z_2) + c_m G_m^{(m)}(z_1, z_2) \tag{8}$$

The coefficients of the defined 2D transfer functions, such as $G_p^{(p)}(z_1, z_2)$ and $D_m(z_1, z_2)$, can be reordered into one-dimensional vectors of length $N_1 \cdot N_2 = M + 1$. This is accomplished by using the ordering arrangement for the support region as given in Fig. 1 b). We define the one-dimensional coefficient vector for $g_p^{(p)}(n_1, n_2)$ as $\mathbf{g}_p^{(p)}$, the coefficient vector for $d_m(n_1, n_2)$ as $\mathbf{d}_m$ and the coefficient vector for $b(n_1, n_2)$ as $\mathbf{b}$. After these definitions, (8) can be rewritten as,

$$\mathbf{d}_{m-1} = \mathbf{d}_m - c_m\, \mathbf{g}_m^{(m)} \tag{9}$$

The backward prediction error transfer functions have the property that $\mathbf{g}_p^{(p)}(p+1) = 1$ for all $p = 0, 1, \ldots, M$ [3]. Therefore, the ladder parameters c_p can be determined by noting that

$$c_p = \mathbf{d}_p(p+1) \tag{10}$$

Using (2), (5) and (6), we get the desired starting point for the recursion which will be employed to determine the ladder coefficients.

$$B(z_1, z_2) = \sum_{p=0}^{M} c_p\, G_p^{(p)}(z_1, z_2) = D_M(z_1, z_2) \tag{11}$$

The recursive algorithm for the calculation of the ladder coefficients is developed using (9), (10) and (11). The algorithm to calculate the lattice-ladder ARMA structure coefficients is as follows:

- The 2D transfer function is given.

  $H(z_1, z_2) = \dfrac{B(z_1, z_2)}{A(z_1, z_2)}$

  - Find the lattice reflection coefficients $\Gamma_{f_{p-n}}^{(n)}$ and $\Gamma_{b_p}^{(n)}$ for $1/A(z_1, z_2)$ using the results in [3].
  - Calculate backward prediction error transfer functions $G_p^{(p)}(z_1, z_2)$ (i.e. $\mathbf{g}_p^{(p)}$), for $p = 0, 1, \ldots, M$.

- Recursive algorithm for the calculation of the ladder coefficients:

  - Initialization:

    $D_M(z_1, z_2) = B(z_1, z_2) \Longrightarrow \mathbf{d}_M = \mathbf{b}$
  - for $p = M : 0$
    - $c_p = \mathbf{d}_p(p+1)$
    - $\mathbf{d}_{p-1} = \mathbf{d}_p - c_p\, \mathbf{g}_p^{(p)}$
  - endfor

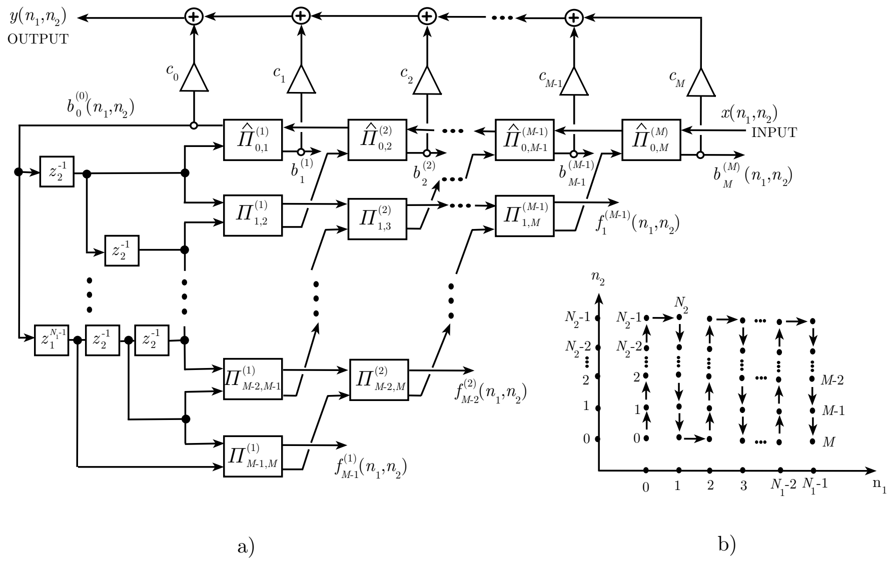

Fig. 1. Lattice-ladder structure; a) Lattice-ladder structure for 2D ARMA filter, b) Ordering scheme in the support region

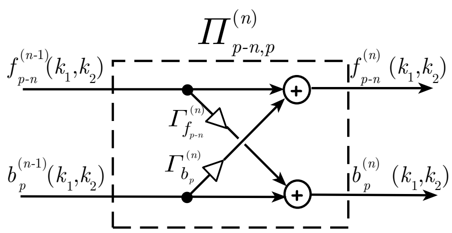

Fig. 2. Internal structure of the FIR lattice module

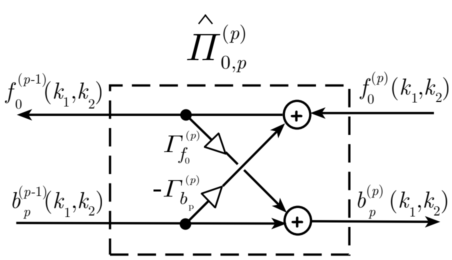

Fig. 3. Internal structure of the IIR lattice module

4. EXAMPLE

We will provide an example for the realization of the 2D ARMA filters using the proposed structure. We assume we are given the following transfer function, which has a support region of $N_1 = N_2 = 2$ with $M = 3$.

$$\begin{aligned} H(z_1, z_2) =& \frac{B(z_1, z_2)}{A(z_1, z_2)} \\ =& \frac{1 + 0.5z_2^{-1} + 0.6z_1^{-1}z_2^{-1} + 0.7z_1^{-1}}{1 + 0.2z_2^{-1} - 0.3z_1^{-1}z_2^{-1} - 0.1z_1^{-1}} \end{aligned} \tag{12}$$

Using the results in [3], the lattice reflection coefficients for the AR section are calculated as follows.

$$\begin{aligned} \Gamma_{f_0}^{(1)} &= \Gamma_{b_1}^{(1)} = 0.1912 \\ \Gamma_{f_1}^{(1)} &= \Gamma_{b_2}^{(1)} = -0.0432 \\ \Gamma_{f_2}^{(1)} &= \Gamma_{b_3}^{(1)} = -0.0001 \\ \Gamma_{f_0}^{(2)} &= \Gamma_{b_2}^{(2)} = -0.2925 \\ \Gamma_{f_1}^{(2)} &= \Gamma_{b_3}^{(2)} = -0.0001 \\ \Gamma_{f_0}^{(3)} &= \Gamma_{b_3}^{(3)} = -0.1000 \end{aligned} \tag{13}$$

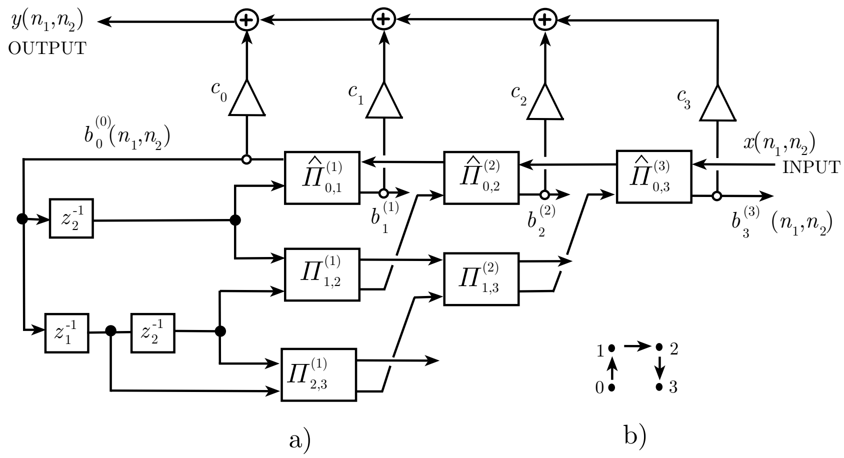

Fig. 4. Lattice-ladder structure for the example; a) Lattice-ladder structure, b) Ordering scheme in the support region for $M = 3$

With these reflection coefficients, we can determine the backward prediction error transfer functions.

$$
\begin{aligned}
G_3^{(3)}(z_1,z_2) &= -0.1000 - 0.0205 z_2^{-1} + 0.0292 z_1^{-1} z_2^{-1} + z_1^{-1} \\
G_2^{(2)}(z_1,z_2) &= -0.2925 - 0.0991 z_2^{-1} + z_1^{-1} z_2^{-1} \\
G_1^{(1)}(z_1,z_2) &= \ 0.1912 + z_2^{-1} \\
G_0^{(0)}(z_1,z_2) &= \ 1
\end{aligned}
\tag{14}
$$

The corresponding coefficient vectors necessary for the calculation of the ladder coefficients are defined as given below.

$$
\begin{aligned}
\mathbf{d}_3 = \mathbf{b} &= \begin{bmatrix} 1 & 0.5 & 0.6 & 0.7 \end{bmatrix} \\
\mathbf{g}_3^{(3)} &= \begin{bmatrix} -0.1000 & -0.0205 & 0.0292 & 1.0000 \end{bmatrix} \\
\mathbf{g}_2^{(2)} &= \begin{bmatrix} -0.2925 & -0.0991 & 1.0000 & 0 \end{bmatrix} \\
\mathbf{g}_1^{(1)} &= \begin{bmatrix} 0.1912 & 1.0000 & 0 & 0 \end{bmatrix} \\
\mathbf{g}_0^{(0)} &= \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix}
\end{aligned}
\tag{15}
$$

When we apply the recursive algorithm for the calculation of the ladder coefficients, we find the ladder coefficients which finalize the lattice-ladder structure.

$$c_0 = 1.1302, \quad c_1 = 0.5718, \quad c_2 = 0.5796, \quad c_3 = 0.7000 \tag{16}$$

It can be easily checked that

$$B(z_1,z_2) = \sum_{p=0}^{3} c_p \, G_p^{(p)}(z_1,z_2) \tag{17}$$

5. CONCLUSIONS

This paper has proposed a novel 2D ARMA lattice-ladder structure. Like the 1D case, the 2D lattice-ladder structure employs linear regression on the backward prediction errors generated by the 2D lattice section. Although there have been previous models for 2D lattice structures, to the best of our knowledge this is the first successful attempt at 2D lattice-ladder filtering. The algorithm to calculate the lattice-ladder structure coefficients for a given 2D ARMA transfer function is included. The 2D lattice-ladder structure maintains the orthogonality and modularity properties of its well-known 1D counterpart. The usage of this novel structure in the 2D adaptive filtering applications and comparison with existing structures will be a subject of further study.

6. REFERENCES

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