PROJECT # 3

The Euler equations governing unsteady compressible inviscid flows can be expressed in conservative form as:

$$\frac{\partial Q}{\partial t} + \frac{\partial F_1}{\partial x_1} + \frac{\partial F_2}{\partial x_2} = 0 \tag{1}$$

where

$$Q = \begin{bmatrix} \rho \\ \rho u_1 \\ \rho u_2 \\ E \end{bmatrix} \quad F_1 = \begin{bmatrix} \rho u_1 \\ \rho u_1^2 + p \\ \rho u_1 u_2 \\ u_1(E+p) \end{bmatrix} \quad F_2 = \begin{bmatrix} \rho u_2 \\ \rho u_1 u_2 \\ \rho u_2^2 + p \\ u_2(E+p) \end{bmatrix} \tag{2}$$

and the internal energy of the gas

$$E = \rho e_{int} + \frac{1}{2}\rho(u_1^2 + u_2^2) \tag{3}$$

The Euler equations are not complete without an equation of state. We choose an ideal gas for which

$$e_{int} = \frac{p}{\rho(\gamma - 1)} \tag{4}$$

The initial conditions are $\rho = 1.4$, $u = 3$, $v = 0$ and $p = 1$. Use the Euler explicit and the fourth-order Runge-Kutta (optional) method with AUSM$^+$-up scheme [1], to solve the forward facing step problem given in Figure 1 at $t = 4$. The state variables Q are defined at the cell centers. The unstructured

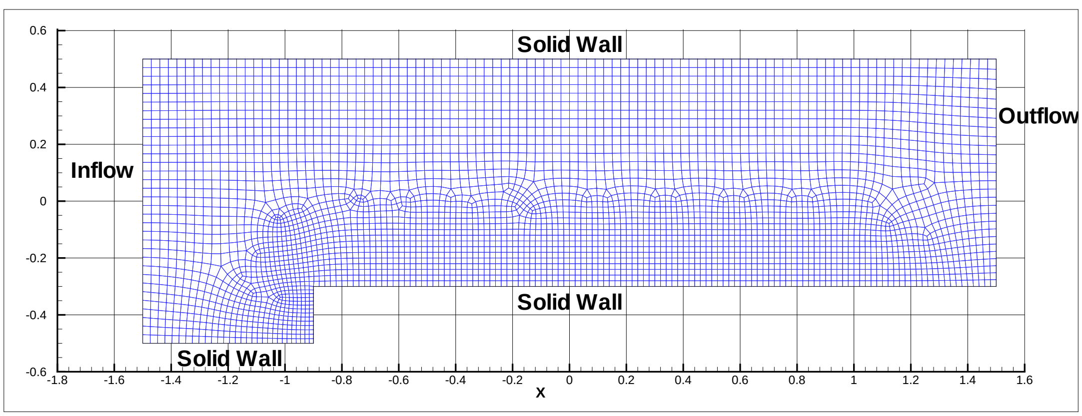

Figure 1: The forward step problem at $M_\infty = 3$ with boundary conditions.

mesh data is available at: `http://web.itu.edu.tr/~msahin/utt619e_2011/forward_step.neu` and the neighbouring element numbers are given at: `http://web.itu.edu.tr/~msahin/utt619e_2011/fort.60`. The zero value indicates that there is no neighbour for that element edge since it is next to the boundary.

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The AUSM$^+$-up Flux

The AUSM$^+$-up for all speeds

$$M_{L/R} = \frac{\mathbf{n} \cdot \mathbf{u}_{L/R}}{a_{1/2}} \tag{5}$$

where $a_{1/2} = (a_L + a_R)/2$

$$\bar{M}^2 = \frac{|\mathbf{u}_L|^2 + |\mathbf{u}_R|^2}{2a_{1/2}^2} \tag{6}$$

$$M_0^2 = min(1, max(\bar{M}^2, M_\infty^2)) \tag{7}$$

$$f_a(M_0) = M_0(2 - M_0) \tag{8}$$

$$\rho_{1/2} = \frac{\rho_L + \rho_R}{2} \tag{9}$$

$$M_{1/2} = M_{(4)}^+(M_L) + M_{(4)}^-(M_R) - \frac{K_p}{f_a} max(1 - \sigma\bar{M}^2, 0)\frac{p_R - p_L}{\rho_{1/2}a_{1/2}^2} \tag{10}$$

where

$$M_{(4)}^\pm(M) = \begin{cases} \frac{1}{2}(M \pm |M|) & \text{if} \quad |M| \geq 1 \\ \pm\frac{1}{4}(M \pm 1)^2(1 + 16\beta\frac{1}{4}(M \mp 1)^2) & \text{otherwise} \end{cases} \tag{11}$$

and $\rho_{1/2} = (\rho_L + \rho_R)/2$, $K_p = 0.25$ and $\sigma = 1$. Then the mass flux

$$\dot{m}_{1/2} = \begin{cases} a_{1/2}M_{1/2}\rho_L & \text{if} \quad M_{1/2} > 0 \\ a_{1/2}M_{1/2}\rho_R & \text{otherwise} \end{cases} \tag{12}$$

the pressure flux

$$p_{1/2} = P_{(5)}^+(M_L)p_L + P_{(5)}^-(M_R)p_R - K_u P_{(5)}^+(M_L)P_{(5)}^-(M_R)(\rho_L + \rho_R)(f_a a_{1/2}^2)(M_R - M_L) \tag{13}$$

and

$$P_{(5)}^\pm(M) = \begin{cases} \frac{1}{2M}(M \pm |M|) & \text{if} \quad |M| \geq 1 \\ \pm\frac{1}{4}(M \pm 1)^2 \left[(\pm 2 - M) + 16\alpha M\frac{1}{4}(M \mp 1)^2\right] & \text{otherwise} \end{cases} \tag{14}$$

using the parameters

$$\alpha = \frac{3}{16}(-4 + 5f_a^2) \qquad \beta = \frac{1}{8} \qquad K_u = 0.75 \tag{15}$$

The whole flux

$$\mathbf{n} \cdot \mathbf{F} = \dot{m}_{1/2} \begin{bmatrix} 1 \\ u_1 \\ u_2 \\ H \end{bmatrix}_{\substack{L \quad \text{if} \quad M_{1/2} > 0 \\ R \quad \text{otherwise}}} + \begin{bmatrix} 0 \\ p_{1/2}n_1 \\ p_{1/2}n_2 \\ 0 \end{bmatrix} \tag{16}$$

where the local enthalpy is given by

$$H = \frac{E + p}{\rho} \tag{17}$$

References

[1] Meng-Sing Liou, A sequel to AUSM, Part II: AUSM$^+$-up for all speeds. *J. Comput. Phys.* **214**, (2006), 137–170.

```
Read Mesh Vetices
for i ← 1 to np do
|   read x[i],y[i],z[i]
end

Read Element Connectivity
for i ← 1 to ne do
|   read nec[i,1],nec[i,2],nec[i,3],nec[i,4]
end

Read Neighbouring Element Numbers
for i ← 1 to ne do
|   read fec[i,1],fec[i,2],fec[i,3],fec[i,4]
end

Initial Q Value
Q^1 := Q_∞


Time integration
for Time ← 0 to 4 do
|
|   Compute Q^{n+1}
|   for i ← 1 to ne do
|   |   RHS := 0
|   |   for n ← 1 to 4 do
|   |   |   ....
|   |   |   ....
|   |   |   ....
|   |   |   if  Solid Wall Boundary Condition then
|   |   |   |   ...
|   |   |   |   RHS := RHS + ...
|   |   |   else if  Inflow Boundary Condition then
|   |   |   |   ...
|   |   |   |   RHS := RHS + ...
|   |   |   else if  Outflow Boundary Condition then
|   |   |   |   ....
|   |   |   |   RHS := RHS + ...
|   |   |   else
|   |   |   |   Q_L = ...
|   |   |   |   Q_R = ...
|   |   |   |   n_x = ...
|   |   |   |   n_x = ...
|   |   |   |   call AUSM_Flux(Q_L, Q_R, n_x, n_y, n · F)
|   |   |   |   RHS := RHS + n · F...
|   |   |   end
|   |   |   ΔQ :=
|   |   end
|   |   Q^{n+1} := Q^n + ΔQ
|   |
|   end
end
```

Table 1: The structure of an unstructured FVM code.